Modeling Heat Flow in a Thermos

AN HONORS THESIS (HONRS 499)

by

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Abstract

The purposes of this thesis are to demonstrate several powerful mathematical techniques such as *Fourier Series* and the *Method of Separation of Variables* by exploiting a simple problem and to illustrate an example that should be accessible to most students who are familiar with second and third semester calculus. A mathematical formula (called the *heat equation*) which describes temperature in a rod will be derived from general principles. This heat equation will then be used to model a simple physical system, namely a thermos filled with ice-cold lemonade. Finally, the correctness of the mathematical model will be experimentally determined. The experiments conducted will test the correctness of the model both within and beyond the constraints of the simplifying assumptions made to derive the model.

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1. Introduction

When introduced to differential equations, one problem that is typically posed to students is the "hot coffee" or "Newton's Coffee Cup" problem. This problem is an introduction to *Newton's Law of Cooling* and, as such, provides a simple model to determine the temperature of an object as a function of time if the object is exposed to surroundings at a constant temperature. The cup of hot coffee is considered as a single unit, and the temperature of the coffee is taken to be an average of the temperature throughout the cup. Named for Sir Isaac Newton (1642-1727), Newton's Cooling is the idea that the rate of temperature loss or gain by an object through interaction with its surroundings is proportional to the difference in temperature between the object and its surroundings.

Thanks to Texas Instruments (TI), high school mathematics classes can easily perform this experiment and verify its results. Using a TI CBL, or Computer-Based Laboratory, a TI calculator, and a program available from TI's website (www.ti.com), temperature data can be collected, plotted, and compared to the model that Newton's Law of Cooling predicts. Generally, good results can be obtained using a simple styrofoam cup with a lid and some hot water. (The lid is important to reduce evaporation and convection and ensure that Newton's law of cooling won't predict too high a temperature [3].)

Consider an added level of complexity–what if one wanted to know the temperature at any position within the cup of coffee at any particular moment in time? Though the "hot coffee" problem may seem artificial and this extension may seem even more so, the modeling of heat flow to find both exact solutions and numeric solutions is of particular interest in the fields of engineering, chemistry, and mathematics. As with the first semester calculus level "hot coffee" problem, this extension serves as an accessible example of such a problem and how to approach finding a solution.

The purposes of this thesis are to demonstrate several powerful mathematical techniques by exploiting a similar problem and to illustrate an example that should be accessible to most students who are familiar with second and third semester calculus. A mathematical formula (called the *heat equation*) which describes temperature in a rod will be derived from general principles. This heat equation will then be used to model a simple physical system, namely a thermos filled with ice-cold lemonade. (The switch to cold lemonade rather than the traditional hot coffee will be explained after the derivation of the heat equation.) Finally, the correctness of the mathematical model will be experimentally determined. The experiments conducted will test the correctness of the model both within and beyond the constraints of the simplifying assumptions made to derive the model.

Problems like this often can be solved by methods based on the ideas of *Fourier Series* and the *Method of Separation of Variables*. Charles Fourier (1772-1837) used these techniques to study heat flow. The reader is assumed to have some knowledge of calculus. In particular, familiarity with limits, series, partial derivatives, and differential equations will be beneficial. Exposure to the idea of orthogonal functions would also be helpful, but certainly is not necessary.

2. Derivation of the One-Dimensional Heat Equation

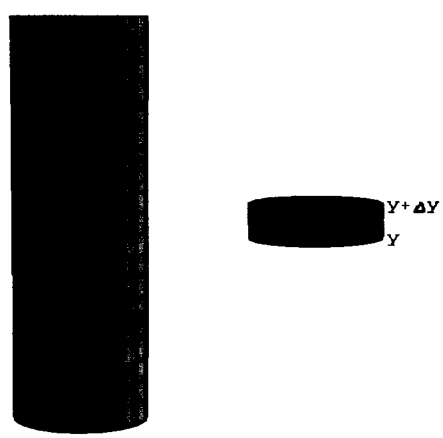
When two regions within a body are at different temperatures, there is naturally a heat flow from the warmer region to the cooler region. The nature of this heat flow can be any or all of three types, [2],

- *Conduction* - passage of heat directly through the material of the body
- *Convection* - passage of heat via motion of the material of the body,
- *Radiation* - passage of heat between possibly distant points of the body by electromagnetic radiation.

In a solid, convection is typically not considered, but conduction and radiation are taken into account. In liquids and gases, all three pathways for heat flow should be considered. If all of the objects being considered are roughly at room temperature, as is the case for us, radiation will play a much lesser role than either conduction or convection.

Consider a long thin rod of conducting material that has an insulated lateral surface and is aligned along the y-axis. Assume that the rod is made of a homogeneous material such that the temperature is the same at any point on a cross section of the rod. This assumption will allow us to think of the temperature in the rod as a function $U[y, t]$ of position y along the y–axis and time t. We wish to find an equation that describes the temperature $U[y, t]$.

First, consider a thin cross-sectional slice of the rod, say from y to $y + \Delta y$. Since *Conservation of Energy* says that we can't magically create or destroy heat, however fast heat enters the slice from the bottom must be as fast as it is either stored in the slice or passes out through the top of the slice. If heat flows in the direction of negative infinity, the heat flow through the surface in the opposite direction is considered negative.



Let $q(y,t)$ be the rate of heat flow per unit area through the slice at position y and time t. The units of q, denoted $[q]$, are $[q] = \frac{H}{tL^2}$ where H, t, and L are heat, time, and length respectively. Let the cross-sectional area of the rod be A, with $[A] = L^2$. Then the heat flow through the bottom of the slice and the top of the slice are given by

$$Aq(y,t)$$

and

$$Aq(y+\Delta y,t),$$

respectively. Since the temperature of the cross-sectional slice is proportional to the amount of heat stored in it, the rate of temperature change will be proportional to the rate of heat storage. Thus, the rate of heat storage in the slice can be approximated by

$$\rho c A \Delta y \frac{\partial U}{\partial t}(y,t),$$

where ρ and c are the *density* and *heat capacity*, respectively. The units are $[\rho] = \frac{m}{L^3}$ (m is mass) and $[c] = \frac{H}{mT}$. Putting it all together, then,

rate of heat in = rate of heat storage + rate of heat out

or

$$Aq(y,t) = \rho c A \Delta y \frac{\partial U}{\partial t}(y,t) + Aq(y+\Delta y,t)$$

Re-arranging and taking the limit as $\Delta y \to 0$ yields

$$\lim_{\Delta y \to 0} \frac{q(y,t) - q(y+\Delta y,t)}{\Delta y} = \frac{\partial q}{\partial y}(y,t) = \rho c \frac{\partial U}{\partial t}(y,t).$$

This looks pretty good, but heat flow rate is difficult to measure directly and what we really want is an equation involving temperature $U[y,t]$ alone. To get another relationship between temperature and heat flow, we now apply *Fourier's Law of Heat Conduction.* This law basically says that heat flows from hot to cold at a rate proportional to how fast the temperature changes spatially. In our notation, Fourier's Law is written

$$q(y,t) = -\kappa \frac{\partial U}{\partial y}(y,t)$$

where κ is a constant known as the *thermal conductivity* of the rod. (Thermal conductivity is a physical property that depends on the specific material.)

Using Fourier's Law and noting that ρ, c, and κ are approximately constant over the temperature range of interest, we obtain the *one-dimensional heat equation (without generation).*

$$\frac{\partial^2 U}{\partial y^2} - \frac{\rho c}{\kappa}\frac{\partial U}{\partial t} = 0.$$

From a fairly simple application of general principles and a couple of physical laws, namely Conservation of Energy and Fourier's Law of Heat Conduction, the one-dimensional heat equation has been derived.

3. The Lemonade Scenario - Modeling $U[y, t]$

3.1. Mathematical Description of System

Consider a cylindrical column of lemonade in a thermos aligned along the y-axis from $y = 0$ to $y = a$. Suppose that this thermos is perfectly insulated on the bottom ($y = 0$) and along the "vertical" surface but allowed to interact via Newton's Cooling with an environment at temperature T_0 at the top ($y = a$). In the physical system or laboratory set-up, this equates to a very good thermos without a lid. This is a minor variation on the model that is commonly used to model objects such as a hot cup of coffee set on a counter to cool because the lid is being left off. From this point forward, "the contents of the thermos", "the thermos", "the fluid", and "the lemonade" or "the column of lemonade" will all be used interchangeably to refer to the cylindrical column of lemonade inside the thermos.

3.2. Initial Value-Boundary Value Problem for System

For simplicity, we will neither consider radiation nor convection in the lemonade. The contribution from radiation should be negligible. Though the convective contribution is likely significant, convection will be assumed to have no impact on the temperature distribution within the thermos due to the added complication that modeling the convection currents would introduce. We choose cold lemonade partially because convective contribution should be less significant for ice-cold lemonade than it would be for a similar set-up involving hot coffee, but primarily to eliminate the problem of evaporative cooling [1].

In addition to assuming no convection, we'll assume that the temperature within each cross-sectional slice is uniform. This is a particularly realistic assumption if, for example, the thermos was well shaken immediately before $t = 0$. Since the lemonade's interaction with its surroundings is only via Newton's Cooling at the top, the temperature within the cross-sectional slice at the top of the column of lemonade will change uniformly through out the slice. From these assumptions, it follows that the temperature within the thermos should be a function only of the vertical position within the thermos and of time.

Let $U[y, t]$ be the temperature of the cross-sectional slice of the fluid at position y and time t and $f[y]$ be the initial temperature distribution. The simple system can then be modeled as follows. (Note that in what follows we will always restrict our model to the intervals $0 < y < a$ and $0 < t < \infty$.)

$$\frac{\partial^2 U}{\partial y^2} - \frac{\rho c}{\kappa}\frac{\partial U}{\partial t} = 0 \tag{1}$$

$$\frac{\partial U}{\partial y}[0, t] = 0 \tag{2}$$

$$U[a, t] + \frac{\kappa}{h}\frac{\partial U}{\partial y}[a, t] = T_0 \tag{3}$$

$$U[y,0] = f[y] \tag{4}$$

Note that (1) is the one-dimensional heat equation. Equation (2) indicates that there is no heat flow through the bottom of the thermos. (According to Fourier's Law of Heat Conduction, this is equivalent to the temperature being constant spatially at the bottom of the thermos.) In a similar way, (3) is a statement indicating that the top of the thermos interacts with its surroundings via Newton's Cooling. Lastly, (4) indicates that the initial temperature distribution is known and is described by some function $f[y]$. Given the nature of the system, this function will necessarily be continuous. We will further assume $f[y]$ to be *sectionally smooth.* (A function $f[y]$ is sectionally smooth on some interval $a \le y \le b$ if $f'[y]$ exists and is continuous, except possibly at a finite number of jumps or removable discontinuities (i.e. $f'[y]$ is *sectionally continuous*) and $f[y]$ is sectionally continuous [6].)

Since the system of equations (1)–(4) includes information about the initial temperature (4) and the way that the temperature changes at the boundaries (2)–(3), this type of problem is referred to as an *Initial-Value Boundary-Value Problem*, or an IV-BVP.

3.3. The Steady-State Solution

To solve (1)–(4), one typically first finds the *steady-state* solution $v[y]$, where $v[y]$ is the limit of U[y,t] as t grows toward infinity. (The steady-state solution can be assumed to exist because of the physical nature of the system. Intuitively, we expect that the contents of the thermos would eventually warm to room temperature, and the temperature would become independent of time. This is why $v[y]$ is called the steady-state solution.) Letting $t \to \infty$ in (1)–(4), we find that the steady-state solution will solve this problem:

$$v''[y] = 0 \tag{5}$$

$$v'[0] = 0 \tag{6}$$

$$v[a] + \frac{\kappa}{h} v'[a] = T_0 \tag{7}$$

Observe the following:

$$\begin{aligned} v''[y] = 0 &\Leftrightarrow v' \text{ is constant} \\ &\Leftrightarrow v'[y] = A \\ &\Leftrightarrow v'[y] = A = 0 \quad \text{(by (6))} \\ &\Leftrightarrow \text{v is constant} \\ &\Leftrightarrow v[y] = B \\ &\Leftrightarrow v[y] = B = T_0 \quad \text{(by (7))} \end{aligned}$$

Thus, $v[y] = T_0$ is the solution to (5)–(7).

3.4. The Transient Solution

Since $U[y,t] \to v[y]$ as $t \to \infty$, if we let $w[y,t] = U[y,t] - w[y]$, it follows that $w[y,t] \to 0$ as $t \to \infty$. For this reason, we call $w[y,t]$ the *transient solution*. Substituting $U[y,t] = w[y,t] + v[y]$ into (1)–(4) produces:

$$\frac{\partial^2}{\partial y^2}(w[y,t] + v[y]) - \frac{\rho c}{\kappa}\frac{\partial}{\partial t}(w[y,t] + v[y]) = 0 \tag{8}$$

$$\frac{\partial}{\partial y}(w[0,t] + v[0]) = 0 \tag{9}$$

$$(w[a,t] + v[a]) + \frac{\kappa}{h}\frac{\partial}{\partial y}(w[a,t] + v[a]) = T_0 \tag{10}$$

$$(w[y,0] + v[y]) = f[y] \tag{11}$$

Substituting $v[y] = T_0$ into (8)–(11) we obtain

$$\frac{\partial^2}{\partial y^2}(w[y,t] + T_0) - \frac{\rho c}{\kappa}\frac{\partial}{\partial t}(w[y,t] + T_0) = 0 \tag{12}$$

$$\frac{\partial}{\partial y}(w[0,t] + T_0) = 0 \tag{13}$$

$$(w[a,t] + T_0) + \frac{\kappa}{h}\frac{\partial}{\partial y}(w[a,t] + T_0) = T_0 \tag{14}$$

$$(w[y,0] + T_0) = f[y] \tag{15}$$

Since the partial derivative of a finite sum is the sum of the partial derivatives of the individual terms of the sum, (12)–(15) yields

$$\frac{\partial^2}{\partial y^2}w[y,t] + \frac{\partial^2}{\partial y^2}T_0 - \frac{\rho c}{\kappa}(\frac{\partial}{\partial t}w[y,t] + \frac{\partial}{\partial t}T_0) = 0 \tag{16}$$

$$\frac{\partial}{\partial y}w[0,t] + \frac{\partial}{\partial y}T_0 = 0 \tag{17}$$

$$(w[a,t] + T_0) + \frac{\kappa}{h}(\frac{\partial}{\partial y}w[a,t] + \frac{\partial}{\partial y}T_0) = T_0 \tag{18}$$

$$(w[y,0] + T_0) = f[y] \tag{19}$$

Observing that the partial derivative of a constant is zero, we obtain from (16)–(19)

$$\frac{\partial^2}{\partial y^2}w[y,t] + 0 - \frac{\rho c}{\kappa}(\frac{\partial}{\partial t}w[y,t] + 0) = 0 \tag{20}$$

$$\frac{\partial}{\partial y}w[0,t] + 0 = 0 \tag{21}$$

$$(w[a,t] + T_0) + \frac{\kappa}{h}(\frac{\partial}{\partial y}w[a,t] + 0) = T_0 \tag{22}$$

$$(w[y,0] + T_0) = f[y] \tag{23}$$

Re-arranging and simplifying (20)–(23) shows that the transient problem is thus defined by

$$\frac{\partial^2 w}{\partial y^2} - \frac{\rho c}{\kappa}\frac{\partial w}{\partial t} = 0 \tag{24}$$

$$\frac{\partial w}{\partial y}[0,t] = 0 \tag{25}$$

$$w[a,t] + \frac{\kappa}{h}\frac{\partial w}{\partial y}[a,t] = 0 \tag{26}$$

$$w[y,0] = g[y] \tag{27}$$

where $g[y] = f[y] - T_0$, or (27) is the initial temperature distribution minus the steady-state solution.

It is significant to note that $w[y,t] = 0$ is a solution to (24)–(26). Because the zero function is a solution to (24)–(26), this system is referred to as *homogeneous*. (Written in this standard form, this is equivalent to the right-hand side, or RHS, of these equations being zero.) Obtaining a homogeneous system of equations is required to prepare for the separation of $w[y,t]$ into a product of spatial and temporal components.

3.5. Separation of Variables

Separation of Variables, or *Fourier's method*, is the name of the technique by which the spatial and temporal components will be separated. Fourier used this method in about 1810 when he made an intensive study of heat conduction problems [6]. The key is to assume that $w[y,t] = \phi[y]T[t]$ where $w[y,t]$ is a solution to the homogeneous system of equations (24)–(26).

Making this assumption, the transient problem (24)–(27) can be re-written as

$$\phi''[y]T[t] - \frac{\rho c}{\kappa}\phi[y]T'[t] = 0 \tag{28}$$

$$\phi'[0]T[t] = 0 \tag{29}$$

$$\phi[a]T[t] + \frac{\kappa}{h}\phi'[a]T[t] = 0 \tag{30}$$

$$\phi[y]T[0] = g[y] \tag{31}$$

By re-arranging (28), we note that

$$\frac{\phi''[y]}{\phi[y]} = \frac{\rho c}{\kappa}\frac{T'[t]}{T[t]} \tag{32}$$

For this to be true, each side of (32) must be constant. To see this, consider fixing y and letting t take all possible values. Namely, let's consider $y = y_0$ and any two arbitrary values of t, t_1 and t_2. Suppose

$$\frac{\phi''[y_0]}{\phi[y_0]} = A_0.$$

Then

$$A_0 = \frac{\rho c}{\kappa}\frac{T'[t_1]}{T[t_1]} \text{ and } A_0 = \frac{\rho c}{\kappa}\frac{T'[t_2]}{T[t_2]}.$$

Thus, the RHS of (32) is constant, with

$$\frac{\rho c}{\kappa}\frac{T'[t]}{T[t]} = A_0 \tag{33}$$

for all t. Using (33), one can see that for some fixed t and allowing y to take all possible values then the left-hand side, or LHS, of (32) must be A_0 for all y. In other words,

$$\frac{\phi''[y]}{\phi[y]} = A_0 = \frac{\rho c}{\kappa}\frac{T'[t]}{T[t]} \tag{34}$$

for all y and t.

From (29) and (30), we obtain the boundary conditions in terms of the product of $T[t]$ and $\phi[y]$ for this particular problem. (29) implies that either $T[t] = 0$ for all t or $\phi'[0] = 0$. Since $T[t] = 0$ for all t forces $w[y, t] = \phi[y]T[t] \equiv 0$ for all y and t (the zero function, or *trivial solution*) and we want our transient solution to be continuous up to the boundary, we require that

$$\phi'[0] = 0. \tag{35}$$

Similarly, (30) implies that either $T[t] = 0$ for all t or

$$\phi[a] + \frac{\kappa}{h}\phi'[a] = 0. \tag{36}$$

To avoid the trivial solution for $w[y, t]$, we discard $T[t] = 0$ and require (36).

Re-arranging (34), we find that $\phi[y]$ and $T[t]$ solve

$$\phi''[y] - A_0\phi[y] = 0, \tag{37}$$

$$T'[t] - A_0\frac{\kappa}{\rho c}T[t] = 0. \tag{38}$$

The solutions to (38) will be of the form

$$T(t) = \xi e^{A_0\frac{\kappa}{\rho c}t},$$

where ξ is a real constant. The solutions to (37) depend on the choice of constant A_0 and boundary conditions (35) and (36). We have three cases to consider: $A_0 > 0$, $A_0 = 0$, and $A_0 < 0$.

Case 1 ($A_0 > 0$): If $A_0 > 0$, the solution to (37) will be of the form

$$\phi(y) = c_1 \cosh[\sqrt{A_0}y] + c_2 \sinh[\sqrt{A_0}y].$$

It follows from (35) that

$$0 = \phi'[0] = c_2\sqrt{A_0},$$

so $c_2 = 0$ must hold. From (36), we see that

$$c_1 \cosh[\sqrt{A_0}a] + c_1\frac{\kappa}{h}\sqrt{A_0}\sinh[\sqrt{A_0}a] = 0 \tag{39}$$

which implies either $c_1 = 0$ or A_0 is a positive solution of (39). In this case we would have a solution to the transient problem (8)–(10) of the form

$$w[y, t] = \xi e^{A_0\frac{\kappa}{\rho c}t}\sinh[\sqrt{A_0}y].$$

Since solutions of this form blow up as $t \to \infty$ for any fixed y, we throw out $A_0 > 0$ as a possibility (the transient solution should decay to zero as $t \to \infty$).

Case 2 ($A_0 = 0$): If $A_0 = 0$, then solutions to (37) will be of the form

$$\phi[y] = c_1 y + c_2.$$

Imposing (35) and (36), we find

$$0 = \phi'[0] = c_1$$

and

$$c_2 + \frac{\kappa}{h}\cdot 0 = 0,$$

so $c_2 = 0$. Again we throw this case out, as we want non-trivial transient solutions.

Case 3 ($A_0 < 0$): For convenience, we take $A_0 = -\lambda^2$. Then (37) has solutions of the form

$$\phi(y) = c_1 \cos[\lambda y] + c_2 \sin[\lambda y].$$

It follows from (35) that

$$0 = \phi'[0] = \lambda c_2,$$

so $c_2 = 0$. From (36), we see that

$$-c_1\lambda\sin[\lambda a] + c_1\frac{h}{k}\cos[\lambda a] = 0$$

which means that either $c_1 = 0$ (leading to trivial $w[y, t]$) or λ must solve

$$\tan[\lambda a] = \frac{h}{\kappa\lambda}. \tag{40}$$

One can show (see Appendix 3) that there are infinitely many solutions λ_n to (40) with one in each interval of the form

$$\frac{(n-1)\pi}{a} < \lambda_n < \frac{(2n-1)\pi}{2a}, \tag{41}$$

where n is a non–zero integer.

Thus, to each positive integer $n = 1, 2, \ldots$ there corresponds a solution

$$\phi_n[y] = B\cos[\lambda_n y],$$

for (35)–(37) with λ_n solving (40), (41) and B an arbitrary real constant. Since the cosine function is *even*, no "new" solutions to (37) are given by negative integers n. Also, for each positive integer n, there is a corresponding solution to (38) of the form

$$T_n(t) = \xi e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}.$$

It follows that for each positive integer n, we have a solution to (8)–(10) of the form

$$w_n[y, t] = \phi_n[y]T_n[t] = B_n \cos[\lambda_n y]e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}. \tag{42}$$

3.6. Sturm-Liouville Problems and Orthogonality

It is interesting to note that the boundary value problem (35)–(37) is a special case of a more general problem known as a *regular Sturm–Liouville* problem. When a function $\phi[y]$ solves this type of problem for a certain λ^2, we call ϕ an *eigenfunction* for problem (35)–(37) with *eigenvalue* λ^2. (Powers' *Boundary Value Problems* [6] provides a nice introduction to the subject of Sturm–Liouville theory. For a more in–depth study of these ideas, see Tolstov's *Fourier Series* [8].) Note that eigenfunctions and eigenvalues for (35)–(37) satisfy the *orthogonality condition*:

$$\int_0^a \phi_n[y]\phi_m[y]\, dy \begin{cases} = 0, & n \neq m, \\ \neq 0, & n = m. \end{cases} \tag{43}$$

To see this, observe that for $n = m$,

$$\begin{aligned} \int_0^a \cos[\lambda_n y]\cos[\lambda_m y] dy &= \frac{1}{2}\left(\frac{\sin[a(\lambda_m - \lambda_n)]}{\lambda_m - \lambda_n} + \frac{\sin[a(\lambda_m + \lambda_n)]}{\lambda_m + \lambda_n}\right) \\ &= \frac{\sin[a(\lambda_m + \lambda_n)]\lambda_m + \sin[a(\lambda_m - \lambda_n)]\lambda_m + \sin[a(\lambda_m - \lambda_n)]\lambda_n - \sin[a(\lambda_m + \lambda_n)]\lambda_n}{2(\lambda_m - \lambda_n)(\lambda_m + \lambda_n)} \\ &= 0. \end{aligned}$$

For $n = m$,

$$\begin{aligned} \int_0^a \cos[\lambda_n y]\cos[\lambda_m y]\, dy &= \int_0^a \cos[\lambda_n y]^2\, dy \\ &= \frac{2a\lambda_n + \sin[2a\lambda_n]}{4\lambda_n} \\ &\neq 0. \end{aligned}$$

3.7. Superposition and the Solution to the Transient Problem

We now know what form solutions to (8)–(10) will take, namely (42). We still need to find a solution to the transient problem that will solve (11). Since each $w_n[y,t] = \phi_n[y]T_n[t]$ is a solution to the homogeneous problem (8)–(10), the *Principle of Superposition* tells us that any finite linear combination of them will also be a solution. We have an infinite family of solutions to (8)–(10), so let's suppose that a solution to (8)–(10) of the form

$$w[y,t] = \sum_{n=1}^{\infty} B_n \cos[\lambda_n y] e^{-\lambda_n{}^2 \frac{\kappa}{\rho c} t} \tag{44}$$

exists and see what conditions on the coefficients B_n must hold for (44) to satisfy (11). Note that the calculations that follow are *formal*, i.e. not mathematically rigorous. (A common technique in mathematics is to argue formally and then prove what one discovers is true rigorously. For the rigorous details that show what follows is valid, see Appendix 4.)

Putting (44) into (11),

$$\begin{aligned} g[y] &= w[y,0] \\ &= \sum_{n=1}^{\infty} \phi_n[y]T_n[0] \\ &= \sum_{n=1}^{\infty} B_n \cos[\lambda_n y] e^{-\lambda^2 \frac{\kappa}{\rho c} 0} \\ &= \sum_{n=1}^{\infty} B_n \cos[\lambda_n y] \end{aligned} \tag{45}$$

Formally, we can use (43) to find the B_n coefficients. Fixing n and multiplying both sides of (45) by $\phi_n[y]$,

$$\begin{aligned} \int_0^a g[y]\cos[\lambda_n y]\,dy &= \int_0^a w[y,0]\cos[\lambda_n y] \\ &= \int_0^a \left(\sum_{m=1}^{\infty} B_m \cos[\lambda_m y]\right)\cos[\lambda_n y]\,dy \\ &= \sum_{m=1}^{\infty} B_m \int_0^a \cos[\lambda_m y]\cos[\lambda_n y]\,dy \\ &= B_n \int_0^a \left(\cos[\lambda_n y]\right)^2 y\,dy \end{aligned} \tag{46}$$

We see then by rearranging (46) that B_n is determined by evaluating

$$B_n = \frac{\int_0^a g[y]\cos[\lambda_n y]\,dy}{\int_0^a (\cos[\lambda_n y])^2\,dy}. \tag{47}$$

Thus, the coefficients for each term of (44) can be determined. The solution to the transient problem (8)–(11) is

$$w[y, t] = \sum_{n=1}^{\infty} B_n \cos[\lambda_n y] e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}$$

where B_n is given by (47) and λ_n is a solution to (40), (41).

3.8. Our Model for the Temperature of the Lemonade in the Thermos

Thus, the solution to the original problem (1)–(4) is

$$u[y, t] = v[y] + w[y, t] = T_0 + \sum_{n=1}^{\infty} B_n \cos[\lambda_n y] e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}. \tag{48}$$

4. Verifying Our Model

Wonderful! We have a model. Unfortunately, a big question still needs to be answered. Namely, how good is our model? There's one easy way to find the answer: test our model.

4.1. Description of the Experimental Procedure

To collect data for to validate the model, we must collect data. The set-up is reasonably simple. The following materials are required:

- Four (4) TI-93 Graphing Calculators (with temperature collection software)
- Four (4) CBLs (with temperature probes)
- Two (2) Twist Ties
- One (1) Rubber Band
- One (1) Thermos
- Ice
- Water
- One (1) Ruler
- One (1) Freezer

The experimental procedure used is as follows:

1. Measure the depth of the interior of the thermos.
2. Using the two twist-ties, connect the temperature probes so that each of the probes are distanced at an interval of one half of the depth of thermos.
3. Place the probes in the freezer.
4. Fill the thermos with ice water and allow it to sit for roughly two hours to pre-chill.
5. After the thermos has been pre-chilled, remove the ice from the thermos and top-off the thermos with ice-cold water.
6. Insert the temperature probes into the thermos and use the rubber band to keep the probes at the proper heights: one at the bottom, one in the middle, and one at the top.
7. Using the TI-85 calculators and the CBLs, record the temperature data. (See Appendix 1 for recorded data for our first experiment.) Temperature data was collected once every 120 seconds for 4 hours.

4.2. Definition of Constants

In order to compare our model to the recorded data, we need to specify the constants in our model. The length of our thermos is $a = 28$cm and the ambient room temperature is measured to be about $T_0 = 21$C. Appropriate values of the density and heat capacity of water are $\rho = 1\ g/cm^3$ and $c = 1$ cal/g-cm, respectively [4]. From Carslaw and Jaeger, *Conduction of Heat in Solids* [2], we find the thermal conductivity of water to be $\kappa = 0.00144 * 60$ cal/cm-sec-C. Based loosely on experimental data, [5], we'll let the convection coefficient $h = 0.002$ cal/cm^2-sec-C.

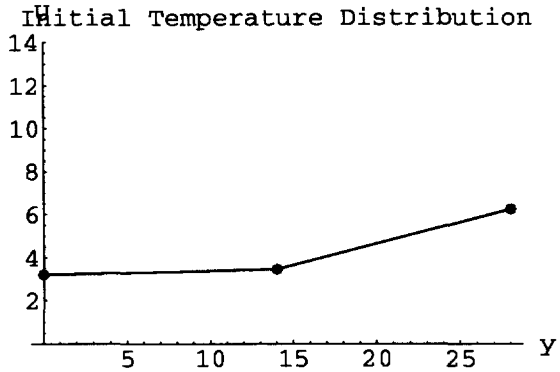
4.3. Definition of $f[y]$

We also need to specify an initial temperature distribution function $f[y]$. Since we expect that the temperature within the thermos will initially be at its coldest, or minimum temperature, at $t = 0$, we will label the initial temperatures at bottom, middle, and top as T_{minC}, T_{minB}, and T_{minA} respectively. If the thermos were well-shaken, we could expect the temperature to be initially constant throughout. In case the initial temperatures at the top, middle, and/or bottom vary slightly, we will use a piecewise linear function for $f[y]$.

Using *Mathematica* to grab the initial temperatures at each height, we construct $f[y]$. (See Appendix 5 for the code used to get the initial temperatures.) Using the points $(0, T_{minC})$, $(a/2, T_{minB})$, and (a, T_{minA}), we construct $f[y]$ by finding equations for the line segments between adjacent points:

$$f[y] := \begin{cases} \left(\frac{T_{minB}-T_{minC}}{\frac{a}{2}}\right)y + T_{minC}, & 0 \le y \le \frac{a}{2}, \\ \left(\frac{T_{minA}-T_{minB}}{\frac{a}{2}}\right)(y - \frac{a}{2}) + T_{minB}, & \frac{a}{2} < y \le a. \end{cases}$$

Here is a plot of $f[y]$ and the initial data points we assumed at the bottom ($y = 0$ cm), middle ($y = a/2 = 14$ cm), and top ($y = 28$ cm) of the thermos.

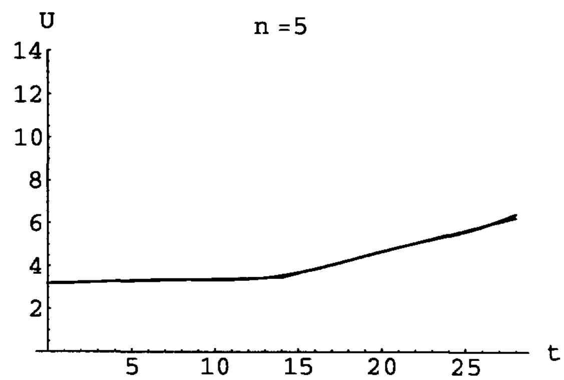


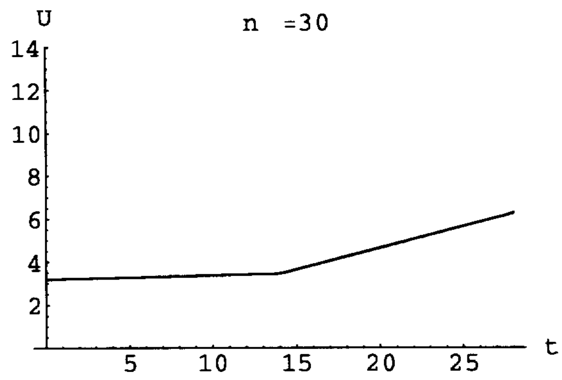
4.4. Obtaining Values for λ_n and B_n

With the specified constant values, we can use the *Bisection Method* to solve (40), (41) for approximate values of λ_n. These λ_n values and the initial temperature distribution $f[y]$ can then be put into (47) to find the B_n coefficients for our model. (See Appendix 5 for the *Mathematica* code that will perform these computations.)

4.5. When is Enough Enough?

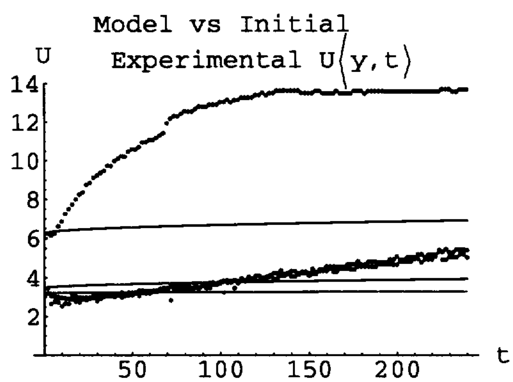
Since our model involves an infinite sum, a natural question arises: "How many terms should we use in our partial sum?" To answer this question, we can use the idea that we want to add in enough terms to ensure that our model well represents the initial temperature distribution $f[y]$. One way to do this is to graphically compare the n^{th} partial sum of (48) at $y = 0$ to the target initial temperature distribution $f[y]$. Here are the graphical comparisons for the cases of five terms and thirty terms in our model. The black curve is $f[y]$ and the red curve is the n^{th} partial sum. It looks like thirty terms in (48) should be enough for our model.





4.6. Behavior of Model

Using thirty terms in (48), we can compare our model to the measured data. In the graph below (and all that follow where we compare our model to recorded temperature data), the dots are recorded data and the smooth curves are model predictions. The red graphs correspond to the top of the thermos ($y = a$), the green graphs correspond to the middle of the thermos ($y = a/2$), and the blue graphs correspond to the bottom of the thermos ($y = 0$).

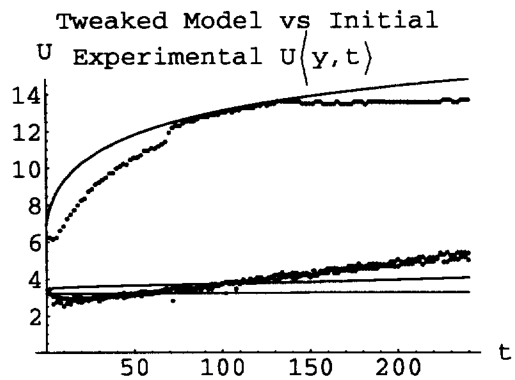


4.7. The Bottom Line

Inspecting our solution visually, the model doesn't seem to match the experimental result very well.

4.8. Tweaking the Model: Re-Definition of the Constant h

Perhaps our model is still ok. Recalling that we chose h fairly arbitrarily, let's select a new value for h. Leaving the other constants the same, we choose $h = 0.002 * 11$ and re-compute the first thirty λ_n and B_n values. The next graph shows how the tweaked model compares to the actual data.



4.9. The Bottom Line for the Tweaked Model

Inspecting our solution visually, the model almost seems to match the experimental results for the temperature at the top of the thermos. Unfortunately, the fit is not very good for the top and is awful for the middle and bottom of the thermos.

5. Verifying Our Model with New Data

Perhaps our assumption that convection plays no significant role is incorrect. To reduce the contribution of this pathway for heat flow, we can make the following modification to our experimental procedure: pack the thermos with cotton balls to (hopefully) reduce convection currents.

5.1. Description of the New Experimental Procedure

The following materials are required for the revised experimental procedure:

- Four (4) TI-93 Graphing Calculators (with temperature collection software)
- Four (4) CBLs (with temperature probes)
- Two (2) Twist Ties
- One (1) Rubber Band
- One (1) Thermos
- One (1) Bag of 100 Cotton Balls
- One (1) Pencil
- Ice
- Water

- One (1) Ruler
- One (1) Freezer

The modified experimental procedure used is as follows:

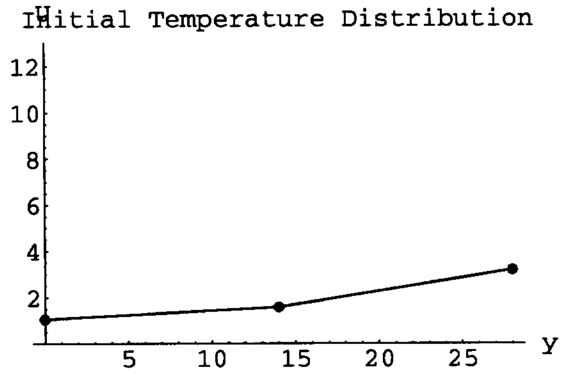
1. Measure the depth of the interior of the thermos.
2. Using the two twist-ties, connect the temperature probes so that each of the probes are distanced at an interval of one half of the depth of thermos.
3. Place the probes, cotton balls, and pencil in the freezer.
4. Fill the thermos with ice water and allow it to sit for roughly two hours to pre-chill.
5. After the thermos has been pre-chilled, remove the ice from the thermos and top-off the thermos with ice-cold water.
6. Insert the temperature probes into the thermos and use the rubber band to keep the probes at the proper heights: one at the bottom, one in the middle, and one at the top.
7. Measure roughly a volume of cotton balls equal to the volume of the thermos and add the cotton balls one-by-one to the thermos. (Use the chilled pencil to poke them below the waterline.)
8. Using the TI-85 calculators and the CBLs, record the temperature data. (See Appendix 2 for recorded data for our new experiment.) Temperature data was collected once every 120 seconds for 4 hours.

5.2. Definition of New Constants

Now that we've adjusted the experimental procedure, we can re-test our model. The same values of $a = 28$, $\rho = 1$, and $c = 1$ are used. The ambient temperature for this data run is slightly higher, so we take $T_0 = 24$ C. Since the cotton, probe leads, and convection could all have an effect on the apparent value of κ and our choice for h was a guess based on experimental results, we tweak κ and h, choosing $\kappa = 0.00144 * 60 * 3$ and $h = 0.00103 * 23$.

5.3. Definition of a New $f[y]$

Using these new constant values, we construct an initial temperature distribution $f[y]$ in the same way as we did first the first experiment. (See Appendix 5.) Here is a graph of $f[y]$ with the new data.

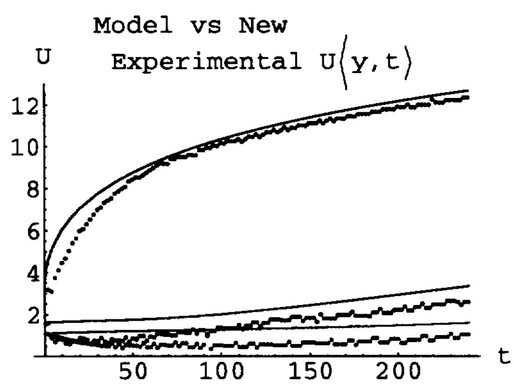


5.4. Obtaining New Values for λ_n and B_n

Just as in the first experiment, we use Mathematica to construct the λ_n and B_n values for our model. (See Appendix 5 for the code we used for this computation.)

5.5. Behavior of Model with New Data

Again, we use thirty terms in (48) for our model. The following plot shows how our model compares to the new measured data.

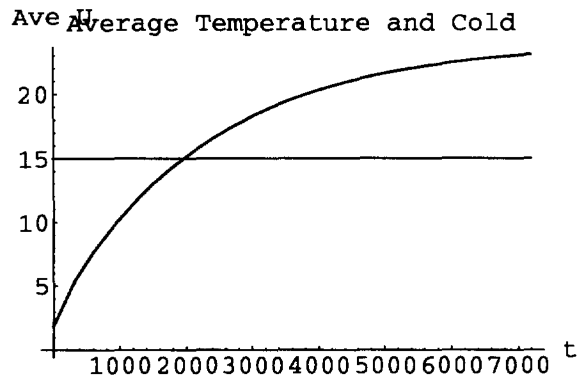


5.6. The Bottom Line with New Data

Inspecting our solution visually, the model seems to match the experimental result reasonably well. The question remains, "How long does the thermos keep things cold?" We can answer this question by looking at the mean temperature of the lemonade, which is the function of time t given by the integral

$$\frac{1}{a}\int_0^a U[y, t]\, dy. \tag{49}$$

This integral needs to be computed numerically, and since it requires a great deal of computation, we take only the first ten terms in (48).



Using *Mathematica*'s "FindRoot" program, we find that the model predicts that the temperature of the thermos will warm up to 15 C in about 1.36466 days.

Our thermos was pretty good. After a day and a half, the lemonade was determined experimentally to still be nearly cold (~16C), so I don't think that it's too unreasonable to expect the lemonade to be cold for a little under a day and a half. It looks like the model works pretty well!

5.7. Definition of an Alternate $f[y]$

Suppose that we had started with some different initial temperature distribution, namely something of the form $f[y] = Ae^{By} + C$. Additionally, since we expect the overall temperature of the contents of the thermos will increase, we choose the "initial" temperatures to which we will fit $f[y]$ to be the minimum temperatures obtained during the course of the run at each of the three heights.

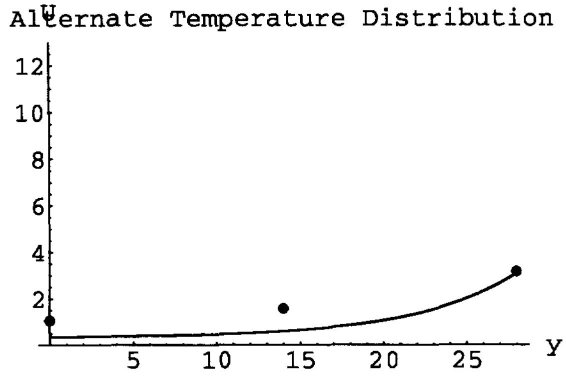
Using *Mathematica* to grab the the minimum temperatures at each height during the course of the run, we find the minimum temperatures T_{minC}, T_{minB}, and T_{minA} at the bottom, middle, and top of the thermos, respectively. To find the unknown constants A, B, and C, we solve the system:

$$f[0] = T_{minC};\ f[a/2] = T_{minB};\ f[a] = T_{minA},$$

yielding

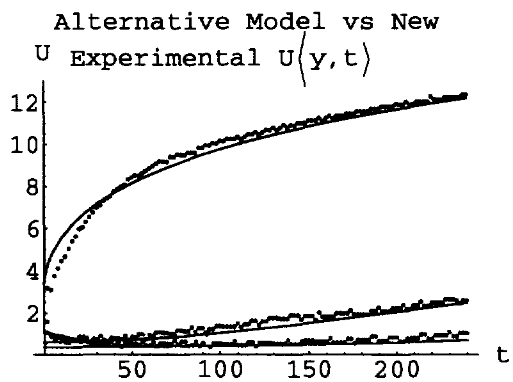
$$f[y] = \left(T_{minC} + \frac{(T_{minB}^2 - T_{minA}T_{minC})}{(T_{minA} - 2T_{minB} + T_{minC})}\right)\exp\left[\left(\frac{2\log\left[\frac{T_{minB}-T_{minA}}{T_{minC}-T_{minB}}\right]}{a}\right)y\right] + \left(\frac{(T_{minA}T_{minC} - T_{minB}^2)}{(T_{minA} - 2T_{minB} + T_{minC})}\right).$$

Here is a graph of $f[y]$ along with the initial temperature values at each height.



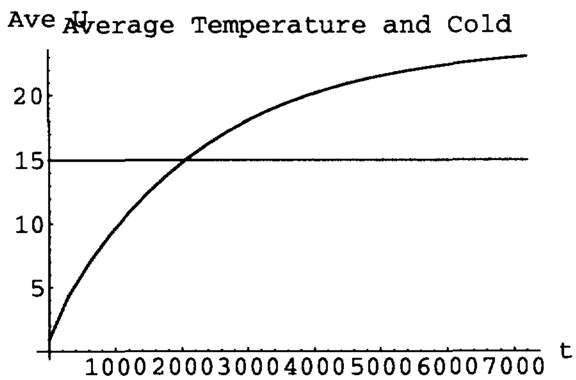
5.8. Behavior of Model with Alternate $f[y]$

As before, we compute the λ_n and B_n values using *Mathematica* and choose h and κ to fit the data. With $\kappa = 0.00144 * 60 * 3$ and $h = 0.00103 * 23$, we get the following results.



5.9. The Bottom Line for the Alternate $f[y]$

Inspecting our solution visually, the model seems to match the experimental result even better than the last experiment. Just as above, we use the mean temperature (49) with the first ten terms of $U[y, t]$ to predict how long the thermos will stay cold. Here is a plot of the mean temperature with the alternate $f[y]$.



Using *Mathematica*'s FindRoot we find that the thermos should stay cold for 1.43018 days. It looks like the model works pretty well... even for a different choice of f[y]!

5.10. The Best Fit

This last model seems to have produced particularly good results. The questions still remaining are "Can we do still better?" and "Why did this last model fit so well?"

Addressing the first question: There is nearly always room for improvement. In particular, the choices of h and κ were made by trial and error. A rough estimate for these values was obtained from the literature initially, and multiples of these values were used when "tweaking" the model. Through minimization of error techniques, the optimal choices for these values and a better fit to the data could both be obtained.

As for the second question: Two significant modifications to the handling of data were made in the last case. First, instead of choosing the initial temperature values as the minimum temperature values, the actual experimental minima were used. This is significant because the mathematical model only allows for the temperature to increase in time. When the initial temperature was not the lowest, we were forced to attempt to overcome this by our choice of h and κ. Second, the choice of $f[y]$ to have the form $f[y] = Ae^{By} + C$ was a significant change from the piecewise linear function that had previously been used. After trying the piecewise linear function, it was considered that the temperature in the thermos might have been constant before the delay associated with the addition of the cotton balls. It seemed that the top of the fluid might have warmed significantly as compared to the temperature at the middle and bottom of the thermos. Examining the experimental data, this possibility was not ruled out, so a function that would provide a gentle increase from the bottom temperature to the middle and then a rapid increase to the temperature at the top of the thermos seemed ideal. An exponential function of the stated form seemed a good choice.

References

[1] J. P. Birk, Coffee Cup Kinetics: A General Chemistry Experiment, *Journal of Chemical Education*, **53 (3)**: 195–196, 1976.

[2] H. J. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, Oxford, 2 edition, 1959.

[3] C. M. Dennis, Jr., Newton's Law of Cooling or Is Ten Minutes Enough Time for a Coffee Break?, *The Physics Teacher*, **18 (10)**: 532–533, 1980.

[4] P. M. Fishbane, S. Gasiorowicz, and S. T. Thorton, *Physics for Scientists and Engineers*, Prentice-Hall, New Jersey, 2 edition, 1996.

[5] H. M. McGee, J. McInerney, and A. Harrus, The Virtual Cook: Modeling Heat Transfer in the Kitchen, *Physics Today*, **52(11)**: 30–36, 1999

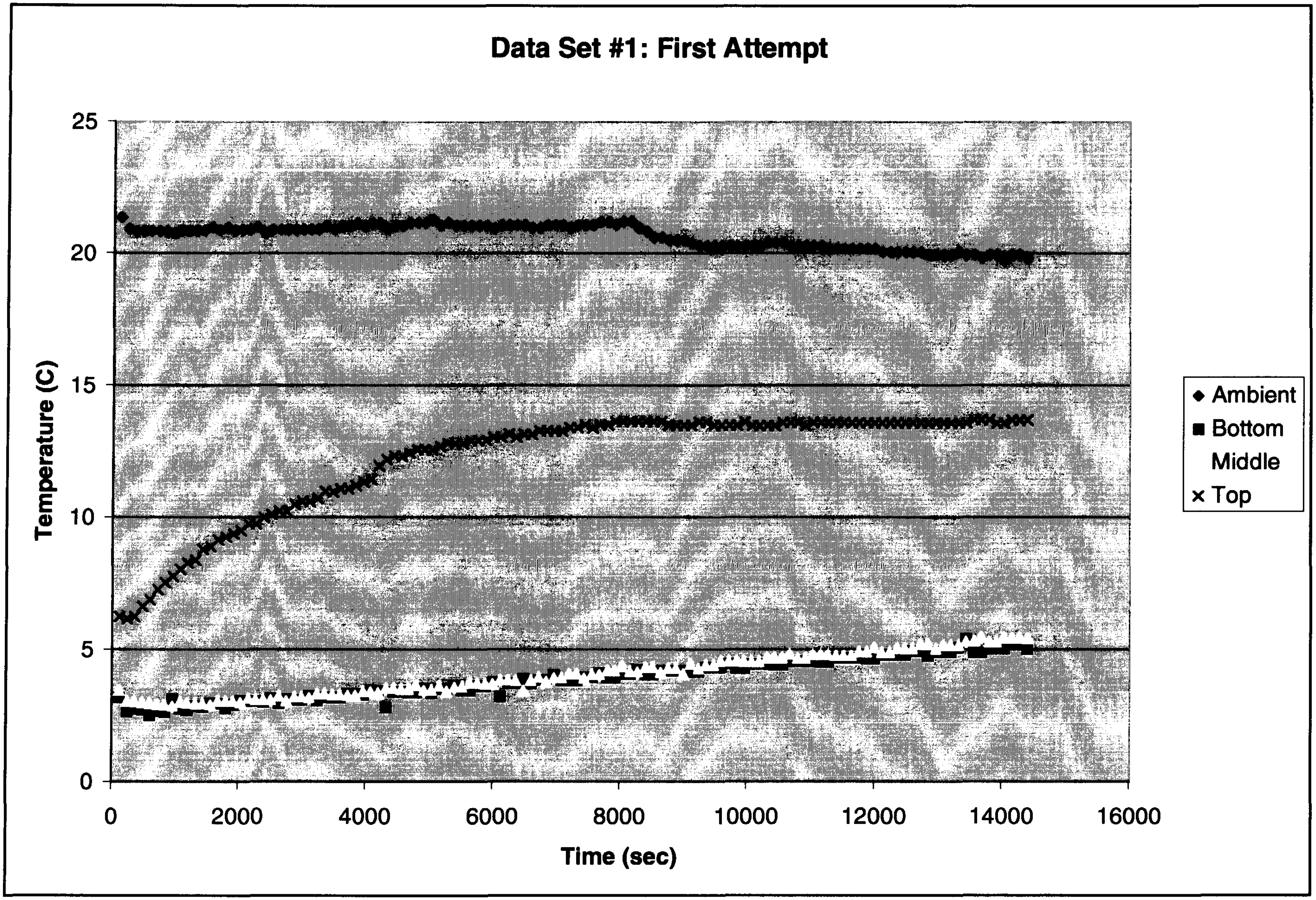
[6] D. L. Powers, *Boundary Value Problems*, Saunders College Publishing, New York, 3 edition, 1987.

[7] W. Rudin, *Principles of Mathematical Analysis*, McGraw–Hill, New York, 3 edition, 1976.

[8] G. P. Tolstov, *Fourier Series*, Prentice Hall, Englewood Cliffs, New Jersey, 1962.

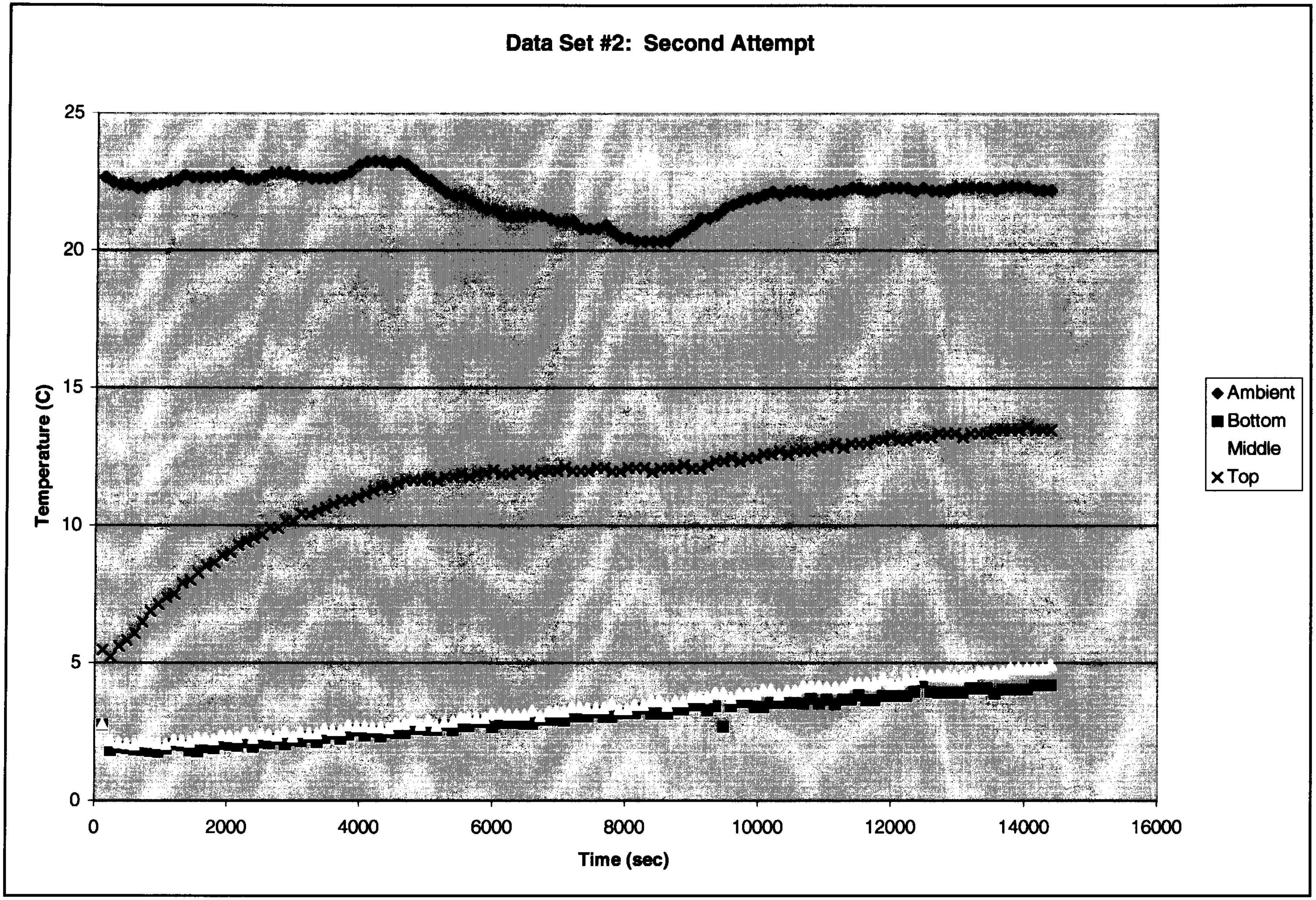
Appendix 1

Only data set 1 was used in the thesis. Data set 2 is included for completeness and, in some small way, to illustrate that the experiment is repeatable.


Data Set #1: First Attempt
Temperature (C)
Time (sec)
0
5
10
15
20
25
0
2000
4000
6000
8000
10000
12000
14000
16000
Ambient
Bottom
Middle
Top

Data Set #1: First Attempt

Time	Ambient	Bottom	Middle	Top	Time	Ambient	Bottom	Middle	Top
120	21.36	3.19	3.45	6.24	7320	21.08	3.83	4.09	13.39
240	20.94	2.65	3.05	6.15	7440	21.08	3.97	3.92	13.51
360	20.83	2.76	3.02	6.24	7560	21.08	3.97	4.06	13.38
480	20.84	2.73	2.99	6.62	7680	21.20	4.12	4.06	13.50
600	20.84	2.52	2.96	6.86	7800	21.20	3.94	4.21	13.50
720	20.85	2.67	2.93	7.26	7920	21.09	3.94	4.21	13.63
840	20.85	2.64	2.90	7.51	8040	21.21	4.09	4.36	13.63
960	20.75	3.13	2.87	7.75	8160	21.21	4.09	4.18	13.62
1080	20.86	2.76	3.01	8.01	8280	20.97	4.24	4.18	13.62
1200	20.86	2.72	2.98	8.26	8400	20.85	4.06	4.33	13.62
1320	20.87	2.87	2.95	8.37	8520	20.62	4.06	4.33	13.62
1440	20.87	2.84	2.95	8.76	8640	20.62	4.21	4.15	13.62
1560	20.98	2.98	2.92	8.87	8760	20.51	4.21	4.30	13.49
1680	20.88	2.95	3.07	9.13	8880	20.51	4.21	4.30	13.49
1800	20.99	2.78	3.04	9.24	9000	20.51	4.21	4.10	13.49
1920	20.88	2.92	3.04	9.37	9120	20.39	4.18	4.45	13.49
2040	20.89	3.07	3.01	9.49	9240	20.28	4.18	4.27	13.61
2160	20.89	3.04	3.16	9.74	9360	20.28	4.33	4.42	13.61
2280	21.00	3.04	3.12	9.73	9480	20.28	4.33	4.42	13.48
2400	20.78	3.01	3.12	9.99	9600	20.28	4.33	4.56	13.48
2520	20.90	3.16	3.09	10.10	9720	20.28	4.47	4.56	13.48
2640	20.90	2.98	3.24	10.23	9840	20.28	4.30	4.56	13.48
2760	20.90	3.12	3.06	10.21	9960	20.28	4.30	4.54	13.61
2880	20.90	3.09	3.21	10.47	10080	20.28	4.45	4.54	13.47
3000	20.90	3.09	3.21	10.59	10200	20.28	4.45	4.54	13.47
3120	20.91	3.24	3.18	10.58	10320	20.40	4.45	4.68	13.47
3240	20.91	3.06	3.33	10.71	10440	20.40	4.42	4.68	13.47
3360	21.03	3.21	3.33	10.96	10560	20.40	4.42	4.68	13.60
3480	20.92	3.18	3.30	10.95	10680	20.29	4.56	4.83	13.60
3600	21.03	3.18	3.30	11.08	10800	20.29	4.56	4.66	13.60
3720	21.03	3.33	3.27	11.07	10920	20.29	4.56	4.66	13.46
3840	21.15	3.33	3.27	11.19	11040	20.29	4.56	4.80	13.59
3960	21.04	3.30	3.41	11.32	11160	20.29	4.54	4.80	13.59
4080	21.15	3.47	3.41	11.43	11280	20.17	4.54	4.80	13.59
4200	21.15	3.44	3.38	11.94	11400	20.17	4.68	4.77	13.59
4320	20.93	2.82	3.53	12.19	11520	20.17	4.68	4.77	13.59
4440	21.04	3.41	3.53	12.32	11640	20.17	4.68	4.77	13.58
4560	21.04	3.41	3.50	12.30	11760	20.17	4.68	4.92	13.58
4680	21.17	3.38	3.50	12.43	11880	20.17	4.66	4.92	13.58
4800	21.17	3.38	3.65	12.56	12000	20.17	4.66	5.07	13.58
4920	21.17	3.53	3.47	12.55	12120	20.06	4.80	4.90	13.58
5040	21.28	3.36	3.47	12.55	12240	20.06	4.80	4.90	13.58
5160	21.06	3.50	3.62	12.67	12360	20.06	4.80	5.04	13.57
5280	21.17	3.50	3.44	12.80	12480	20.06	4.80	5.04	13.57
5400	21.06	3.65	3.59	12.79	12600	20.06	4.95	5.04	13.57
5520	21.06	3.47	3.59	12.79	12720	20.06	4.95	5.19	13.57
5640	21.06	3.47	3.74	12.91	12840	19.94	4.77	5.19	13.57
5760	21.06	3.62	3.74	12.91	12960	19.94	4.92	5.01	13.57
5880	21.06	3.62	3.71	12.90	13080	19.94	4.92	5.16	13.57
6000	20.95	3.77	3.71	13.03	13200	19.94	4.92	5.16	13.57
6120	21.07	3.23	3.86	13.03	13320	20.06	5.07	5.16	13.57
6240	21.07	3.74	3.86	13.15	13440	19.94	5.41	5.31	13.57
6360	21.07	3.74	3.83	13.02	13560	19.94	4.90	5.31	13.69
6480	21.07	3.89	3.47	13.14	13680	19.83	4.90	5.45	13.69
6600	20.96	3.71	3.83	13.14	13800	19.95	5.04	5.28	13.69
6720	20.96	3.86	3.97	13.27	13920	19.95	5.04	5.28	13.56
6840	21.08	3.86	3.80	13.27	14040	19.71	5.04	5.43	13.56
6960	21.08	4.03	3.94	13.26	14160	19.95	5.19	5.43	13.68
7080	21.08	3.83	3.94	13.26	14280	19.95	5.19	5.43	13.68
7200	20.96	3.83	4.09	13.39	14400	19.83	5.01	5.40	13.68

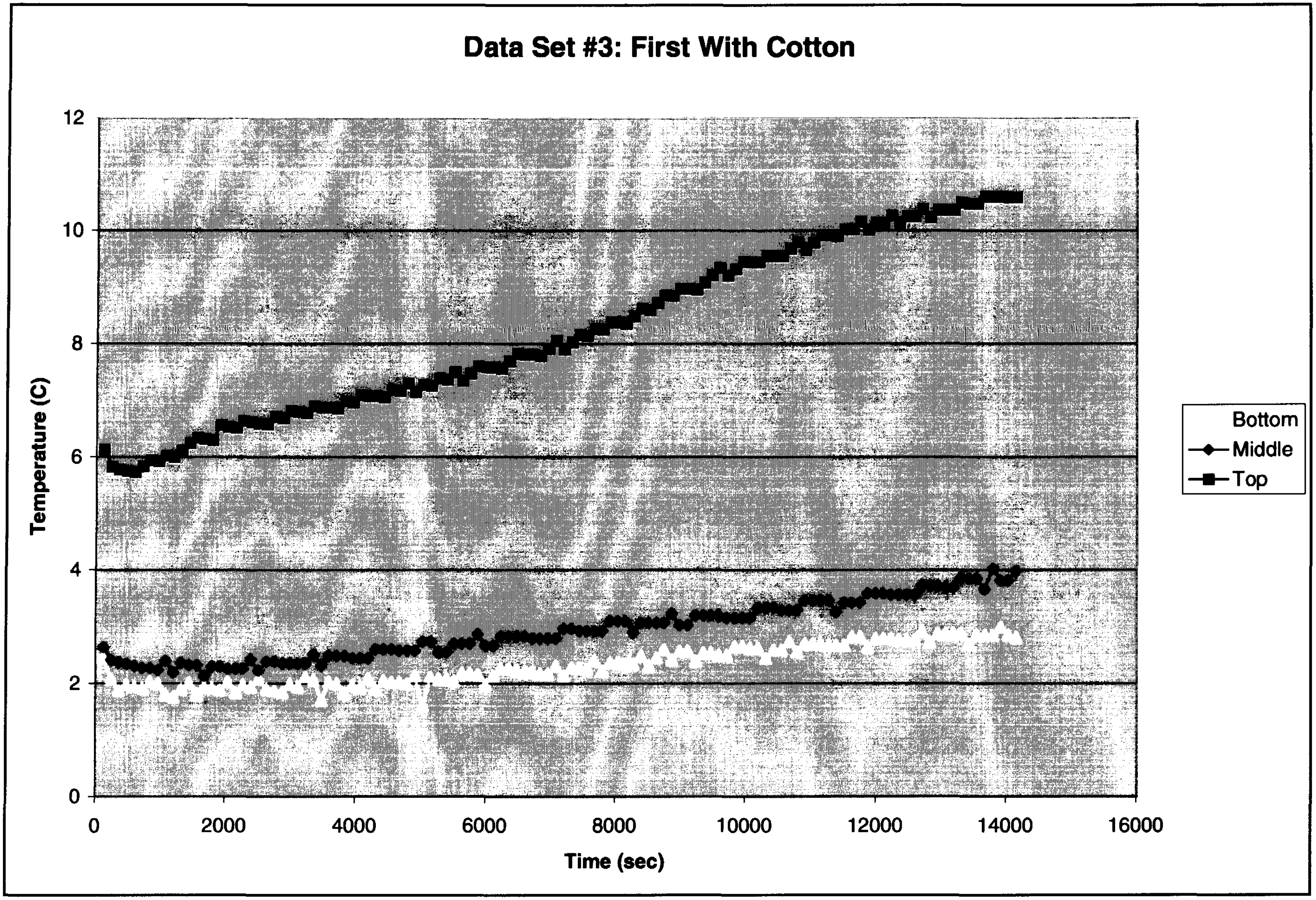

Data Set #2: Second Attempt
Temperature (C)
Time (sec)
25
20
15
10
5
0
0
2000
4000
6000
8000
10000
12000
14000
16000
Ambient
Bottom
Middle
Top

Data Set #2: Second Attempt

Time	Ambient	Bottom	Middle	Top	Time	Ambient	Bottom	Middle	Top
120	22.66	2.76	2.78	5.47	7320	20.81	3.07	3.38	11.98
240	22.47	1.82	2.17	5.21	7440	20.81	3.07	3.35	11.98
360	22.38	1.94	2.10	5.60	7560	20.81	3.22	3.35	12.11
480	22.39	1.91	2.07	5.83	7680	20.93	3.04	3.35	12.11
600	22.28	1.87	2.03	6.08	7800	20.69	3.04	3.51	11.96
720	22.29	1.84	2.15	6.49	7920	20.46	3.19	3.32	11.96
840	22.41	1.81	2.12	6.88	8040	20.46	3.19	3.32	12.10
960	22.42	1.77	2.08	7.13	8160	20.34	3.19	3.47	12.10
1080	22.53	1.89	2.05	7.38	8280	20.34	3.35	3.47	12.10
1200	22.54	2.04	2.20	7.50	8400	20.34	3.16	3.63	11.95
1320	22.77	2.01	2.17	7.90	8520	20.34	3.16	3.63	12.08
1440	22.66	1.82	2.13	8.01	8640	20.34	3.16	3.44	12.08
1560	22.67	1.79	2.29	8.28	8760	20.58	3.31	3.60	12.08
1680	22.68	1.94	2.25	8.53	8880	20.70	3.31	3.60	12.21
1800	22.68	1.91	2.22	8.65	9000	20.94	3.47	3.75	12.07
1920	22.69	2.06	2.37	8.91	9120	21.19	3.47	3.75	12.07
2040	22.81	2.03	2.34	9.03	9240	21.19	3.28	3.75	12.20
2160	22.70	1.99	2.31	9.29	9360	21.31	3.44	3.91	12.34
2280	22.60	2.14	2.46	9.41	9480	21.55	2.69	3.91	12.34
2400	22.60	1.96	2.27	9.54	9600	21.67	3.44	3.72	12.46
2520	22.72	2.11	2.42	9.66	9720	21.79	3.59	3.87	12.32
2640	22.84	2.08	2.39	9.92	9840	21.91	3.59	3.87	12.45
2760	22.85	2.23	2.36	9.91	9960	21.92	3.41	3.87	12.45
2880	22.85	2.04	2.51	10.17	10080	22.04	3.41	4.03	12.59
3000	22.74	2.20	2.32	10.15	10200	22.16	3.56	4.03	12.59
3120	22.74	2.16	2.48	10.42	10320	22.04	3.56	3.84	12.72
3240	22.63	2.32	2.44	10.40	10440	22.16	3.56	4.00	12.57
3360	22.63	2.13	2.44	10.53	10560	22.16	3.56	4.00	12.71
3480	22.64	2.29	2.60	10.65	10680	22.16	3.53	4.15	12.71
3600	22.64	2.44	2.56	10.78	10800	22.05	3.71	4.15	12.71
3720	22.76	2.25	2.56	10.91	10920	22.05	3.53	4.15	12.84
3840	22.89	2.41	2.72	10.90	11040	22.05	3.68	4.12	12.84
3960	23.12	2.56	2.69	11.03	11160	22.17	3.50	4.12	12.97
4080	23.24	2.37	2.69	11.15	11280	22.17	3.68	4.12	12.83
4200	23.25	2.53	2.65	11.28	11400	22.29	3.68	4.27	12.96
4320	23.25	2.34	2.65	11.41	11520	22.29	3.83	4.27	12.96
4440	23.14	2.49	2.62	11.39	11640	22.18	3.65	4.27	12.96
4560	23.26	2.46	2.77	11.52	11760	22.18	3.65	4.27	13.09
4680	23.14	2.46	2.77	11.65	11880	22.30	3.80	4.43	13.09
4800	22.91	2.61	2.93	11.64	12000	22.30	3.80	4.24	13.22
4920	22.68	2.61	2.74	11.64	12120	22.30	3.80	4.24	13.08
5040	22.56	2.58	2.89	11.77	12240	22.30	3.80	4.40	13.08
5160	22.33	2.58	2.71	11.63	12360	22.18	3.96	4.40	13.21
5280	22.10	2.74	2.86	11.76	12480	22.30	4.32	4.55	13.21
5400	21.98	2.55	2.86	11.76	12600	22.19	3.96	4.55	13.21
5520	21.98	2.70	3.02	11.89	12720	22.19	3.93	4.55	13.34
5640	21.86	2.70	3.02	11.75	12840	22.19	3.93	4.55	13.34
5760	21.63	2.86	2.99	11.88	12960	22.31	3.93	4.52	13.34
5880	21.51	2.86	2.99	11.88	13080	22.31	3.93	4.52	13.20
6000	21.51	2.67	3.14	12.01	13200	22.31	4.08	4.52	13.33
6120	21.28	2.83	3.14	11.87	13320	22.31	4.08	4.68	13.33
6240	21.28	2.83	3.11	11.87	13440	22.31	4.08	4.68	13.33
6360	21.28	2.83	3.11	12.00	13560	22.20	3.90	4.68	13.46
6480	21.28	2.79	3.11	12.00	13680	22.32	4.05	4.68	13.46
6600	21.28	2.79	3.26	11.86	13800	22.32	4.05	4.83	13.46
6720	21.28	2.95	3.07	11.99	13920	22.32	4.05	4.83	13.46
6840	21.16	2.95	3.23	11.99	14040	22.32	4.05	4.80	13.59
6960	21.05	2.95	3.23	11.99	14160	22.20	4.20	4.80	13.45
7080	21.05	2.91	3.23	12.12	14280	22.20	4.20	4.80	13.45
7200	21.05	3.07	3.38	11.98	14400	22.20	4.20	4.95	13.45

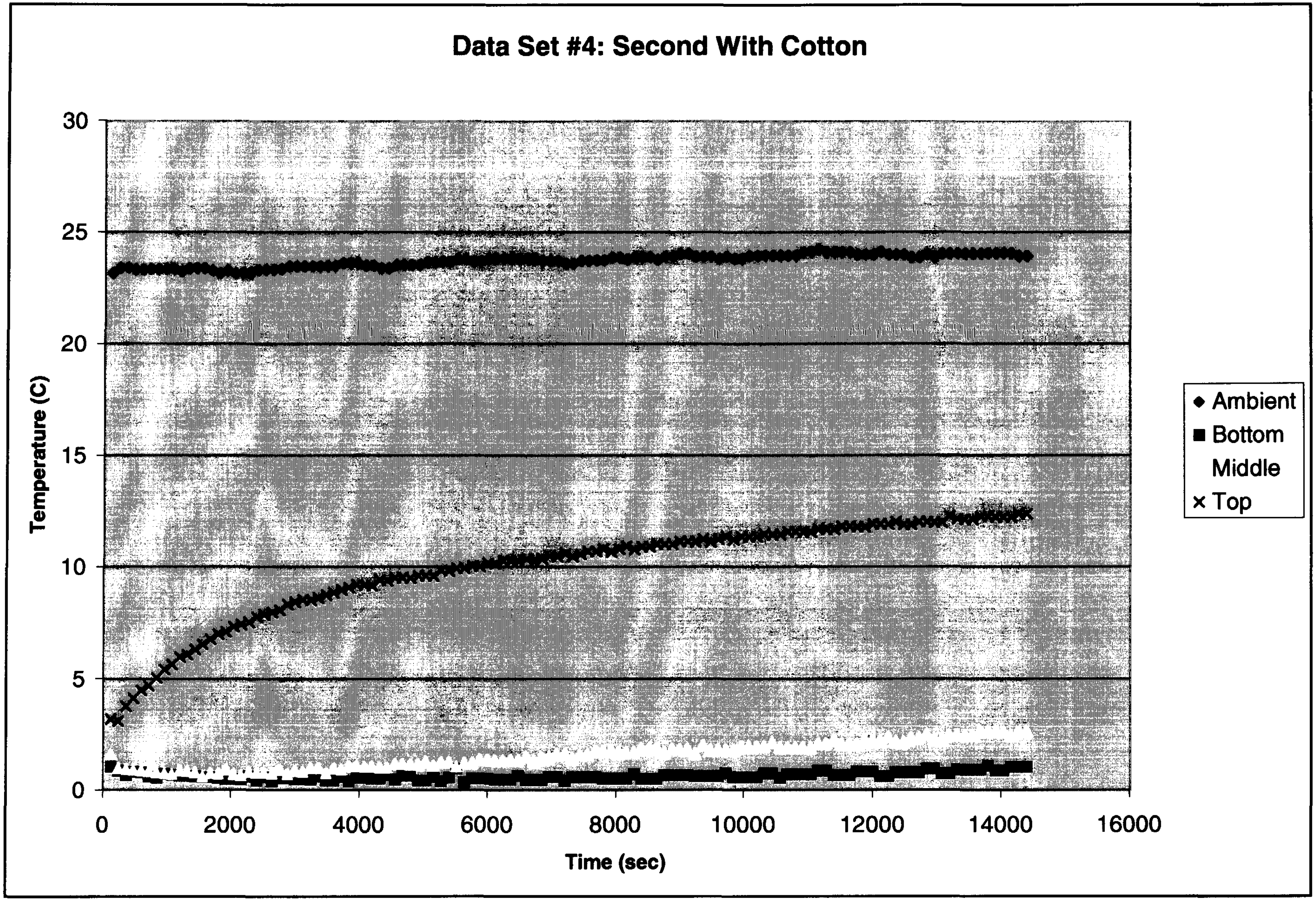
Appendix 2

Only data set 4 was used in the thesis. Data set 3 is included for completeness and, in some small way, to illustrate that the experiment is repeatable. No ambient temperature data is included with data set 3 because the batteries in the CBL went dead during the course of the run.


Data Set #3: First With Cotton
12
10
8
6
4
2
0
Temperature (C)
0
2000
4000
6000
8000
10000
12000
14000
16000
Time (sec)
Bottom
Middle
Top

Data Set #3: First With Cotton

Time	Top	Middle	Bottom	Time	Top	Middle	Bottom
120	6.13	2.64	2.47	7320	8.03	2.97	2.29
240	5.82	2.42	2.08	7440	8.16	2.93	2.29
360	5.77	2.38	1.88	7560	8.14	2.93	2.26
480	5.75	2.35	1.99	7680	8.27	2.93	2.41
600	5.73	2.31	1.96	7800	8.25	2.93	2.23
720	5.83	2.28	1.93	7920	8.38	3.10	2.38
840	5.94	2.28	2.04	8040	8.38	3.10	2.38
960	5.92	2.24	2.01	8160	8.36	3.10	2.35
1080	6.02	2.40	1.81	8280	8.49	2.90	2.35
1200	6.00	2.20	1.77	8400	8.62	3.07	2.49
1320	6.11	2.37	1.91	8520	8.60	3.07	2.31
1440	6.24	2.33	2.02	8640	8.73	3.07	2.46
1560	6.34	2.33	1.82	8760	8.86	3.07	2.61
1680	6.32	2.13	1.96	8880	8.85	3.23	2.43
1800	6.30	2.30	1.93	9000	8.97	3.03	2.58
1920	6.56	2.30	1.90	9120	8.97	3.03	2.58
2040	6.53	2.26	2.04	9240	8.96	3.20	2.40
2160	6.51	2.26	1.84	9360	9.08	3.20	2.55
2280	6.64	2.26	1.98	9480	9.21	3.20	2.55
2400	6.62	2.43	1.95	9600	9.34	3.20	2.51
2520	6.60	2.23	2.09	9720	9.20	3.16	2.51
2640	6.58	2.39	1.92	9840	9.32	3.16	2.66
2760	6.71	2.39	1.88	9960	9.45	3.16	2.66
2880	6.69	2.36	1.85	10080	9.44	3.16	2.63
3000	6.82	2.36	2.00	10200	9.44	3.33	2.63
3120	6.80	2.36	1.96	10320	9.56	3.33	2.45
3240	6.78	2.36	2.11	10440	9.55	3.33	2.60
3360	6.90	2.52	2.07	10560	9.55	3.29	2.60
3480	6.88	2.32	1.72	10680	9.68	3.29	2.75
3600	6.88	2.49	2.04	10800	9.80	3.29	2.57
3720	6.86	2.49	2.01	10920	9.66	3.46	2.71
3840	6.99	2.49	1.84	11040	9.79	3.46	2.71
3960	6.97	2.45	1.98	11160	9.92	3.46	2.71
4080	7.10	2.45	1.95	11280	9.92	3.46	2.68
4200	7.08	2.45	2.09	11400	9.90	3.26	2.68
4320	7.08	2.61	1.92	11520	10.03	3.42	2.68
4440	7.06	2.61	2.06	11640	10.03	3.42	2.83
4560	7.19	2.61	2.03	11760	10.16	3.42	2.83
4680	7.17	2.58	2.03	11880	10.02	3.59	2.65
4800	7.30	2.58	2.00	12000	10.14	3.59	2.80
4920	7.15	2.58	2.14	12120	10.14	3.59	2.80
5040	7.28	2.74	1.61	12240	10.27	3.56	2.80
5160	7.26	2.74	2.11	12360	10.13	3.56	2.77
5280	7.39	2.55	2.08	12480	10.26	3.56	2.77
5400	7.37	2.55	2.08	12600	10.26	3.56	2.77
5520	7.50	2.71	2.05	12720	10.38	3.72	2.92
5640	7.35	2.71	2.19	12840	10.24	3.72	2.74
5760	7.48	2.71	2.19	12960	10.37	3.72	2.89
5880	7.61	2.87	2.16	13080	10.37	3.69	2.89
6000	7.59	2.67	1.98	13200	10.37	3.69	2.89
6120	7.59	2.67	2.13	13320	10.50	3.85	2.89
6240	7.57	2.84	2.27	13440	10.48	3.85	2.71
6360	7.70	2.84	2.27	13560	10.48	3.85	2.86
6480	7.83	2.84	2.24	13680	10.61	3.66	2.86
6600	7.81	2.84	2.21	13800	10.61	4.02	2.86
6720	7.81	2.80	2.21	13920	10.61	3.82	3.00
6840	7.79	2.80	2.18	14040	10.59	3.82	2.86
6960	7.92	2.80	2.18	14160	10.59	3.98	2.82
7080	8.05	2.80	2.32	14280	10.72	3.98	2.82
7200	7.90	2.97	2.15	14400	10.72	3.98	2.97



Data Set #4: Second With Cotton
30
25
20
15
10
5
0
Temperature (C)
0
2000
4000
6000
8000
10000
12000
14000
16000
Time (sec)
Ambient
Bottom
Middle
Top

Data Set #4: Second With Cotton

Time	Ambient	Bottom	Middle	Top	Time	Ambient	Bottom	Middle	Top
120	23.16	1.06	1.58	3.18	7320	23.59	0.58	1.42	10.45
240	23.38	0.87	0.98	3.09	7440	23.72	0.58	1.59	10.57
360	23.41	0.68	0.94	3.75	7560	23.72	0.58	1.59	10.69
480	23.31	0.84	0.9	4.11	7680	23.74	0.58	1.75	10.68
600	23.32	0.8	0.87	4.47	7800	23.74	0.53	1.75	10.8
720	23.33	0.61	0.83	4.68	7920	23.87	0.53	1.75	10.67
840	23.34	0.57	0.79	5.04	8040	23.87	0.53	1.71	10.79
960	23.35	0.73	0.76	5.4	8160	23.75	0.53	1.71	10.92
1080	23.36	0.69	0.72	5.63	8280	23.88	0.7	1.71	10.78
1200	23.24	0.65	0.88	5.98	8400	23.88	0.5	1.88	10.9
1320	23.37	0.65	0.84	6.07	8520	23.88	0.5	1.88	10.89
1440	23.38	0.62	0.8	6.3	8640	23.76	0.5	1.88	11.01
1560	23.38	0.58	0.6	6.53	8760	23.9	0.66	1.68	11.01
1680	23.27	0.58	0.76	6.76	8880	23.9	0.66	1.84	11
1800	23.15	0.54	0.73	6.99	9000	24.03	0.66	1.84	11.12
1920	23.28	0.5	0.89	7.09	9120	24.03	0.66	1.84	11.12
2040	23.16	0.5	0.69	7.32	9240	23.91	0.62	1.64	11.11
2160	23.16	0.47	0.65	7.43	9360	23.91	0.62	2	11.24
2280	23.17	0.62	0.81	7.53	9480	23.91	0.62	1.81	11.1
2400	23.3	0.43	0.77	7.76	9600	23.78	0.62	1.81	11.23
2520	23.31	0.59	0.77	7.87	9720	23.91	0.78	1.81	11.35
2640	23.31	0.39	0.74	7.98	9840	23.79	0.58	1.81	11.21
2760	23.33	0.55	0.9	8.08	9960	23.79	0.58	1.97	11.34
2880	23.46	0.55	0.7	8.33	10080	23.92	0.58	1.97	11.34
3000	23.46	0.51	0.86	8.43	10200	23.92	0.58	1.97	11.33
3120	23.47	0.51	0.82	8.54	10320	23.94	0.75	1.97	11.45
3240	23.47	0.47	0.98	8.52	10440	23.94	0.75	2.13	11.45
3360	23.47	0.63	0.98	8.63	10560	23.94	0.55	1.93	11.44
3480	23.48	0.44	0.75	8.74	10680	23.95	0.71	1.93	11.56
3600	23.48	0.44	0.95	8.86	10800	23.95	0.71	1.93	11.56
3720	23.61	0.6	0.91	8.97	10920	24.09	0.71	1.93	11.55
3840	23.62	0.4	1.07	9.09	11040	24.09	0.71	1.9	11.55
3960	23.62	0.56	1.07	9.2	11160	24.22	0.87	2.09	11.67
4080	23.51	0.56	1.04	9.18	11280	24.1	0.87	2.09	11.66
4200	23.51	0.52	1.04	9.17	11400	24.1	0.67	2.26	11.66
4320	23.38	0.52	1	9.41	11520	24.1	0.67	2.06	11.79
4440	23.38	0.68	1	9.4	11640	24.1	0.67	2.06	11.79
4560	23.52	0.48	1.16	9.52	11760	23.98	0.83	2.06	11.78
4680	23.52	0.64	1.16	9.51	11880	23.98	0.83	2.22	11.78
4800	23.52	0.64	1.12	9.49	12000	23.98	0.83	2.22	11.9
4920	23.53	0.44	1.12	9.61	12120	24.11	0.63	2.22	11.89
5040	23.66	0.44	1.28	9.6	12240	23.99	0.63	2.38	11.89
5160	23.66	0.6	1.08	9.58	12360	23.99	0.79	2.18	12.01
5280	23.67	0.4	1.25	9.83	12480	23.99	0.79	2.18	11.88
5400	23.67	0.56	1.25	9.82	12600	23.87	0.79	2.35	11.88
5520	23.81	0.96	1.05	9.94	12720	23.87	0.79	2.35	12
5640	23.81	0.36	1.21	9.93	12840	24.01	0.96	2.51	12
5760	23.69	0.53	1.21	10.05	12960	23.88	0.96	2.31	11.99
5880	23.69	0.53	1.37	10.03	13080	24.02	0.76	2.31	11.99
6000	23.82	0.53	1.37	10.16	13200	24.02	0.76	2.47	12.25
6120	23.82	0.49	1.34	10.14	13320	24.02	0.92	2.47	12.12
6240	23.82	0.49	1.34	10.27	13440	24.03	0.92	2.47	12.11
6360	23.82	0.49	1.34	10.25	13560	24.03	0.92	2.47	12.11
6480	23.83	0.65	1.34	10.25	13680	24.03	0.92	2.64	12.23
6600	23.83	0.45	1.5	10.36	13800	24.05	1.08	2.64	12.23
6720	23.83	0.45	1.3	10.23	13920	24.05	0.88	2.64	12.22
6840	23.71	0.45	1.46	10.35	14040	24.05	0.88	2.6	12.22
6960	23.71	0.61	1.46	10.47	14160	24.05	1.05	2.44	12.22
7080	23.71	0.61	1.62	10.46	14280	23.92	1.05	2.6	12.35
7200	23.59	0.41	1.62	10.58	14400	23.92	1.05	2.6	12.35

Appendix 3

Theorem: *The function*

$$\tan[\lambda_n a] = \frac{h}{\kappa \lambda_n} \tag{1}$$

has infinitely many solutions ($n = 1, 2, 3, \ldots$) such that $\dfrac{(n-1)\pi}{a} < \lambda_n < \dfrac{(2n-1)\pi}{2a}$.

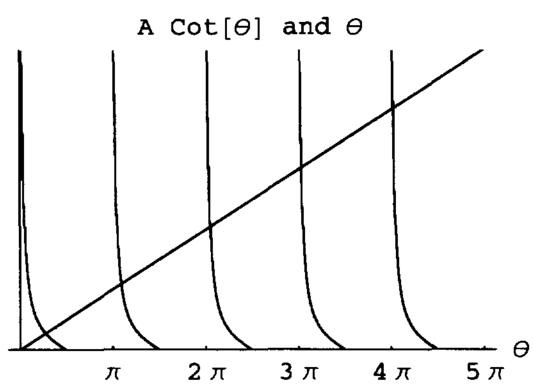
Proof: Restrict $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots.$ Observe that h, a, and κ are positive quantities, so we can re-arrange (1) to obtain

$$\frac{ha}{\kappa} \cot \lambda_n a = \lambda_n a \tag{2}$$

Let's replace $\lambda_n a$ by θ and $\dfrac{ha}{\kappa}$ by A ($A > 0$) to obtain a simpler problem

$$A \cot \theta = \theta \tag{3}$$

Looking at a graph of $A \cot \theta$ and θ as functions of θ,



we are reminded that $\cot \theta$ has a period of π and $\cot \theta \in [0, \infty)$ on the interval $0 < \theta < \pi/2$ so $A \cot \theta$ is positive for $(n-1)\pi < \theta < \dfrac{(2n-1)}{2}\pi$ ($n = 1, 2, 3, \ldots$). Also, observing that $\dfrac{d}{d\theta}(A \cot \theta) = -A \csc^2 \theta < 0$ we see that $A \cot \theta$ is a strictly decreasing function on each of these intervals. Since $y = \theta$ is a non-negative and strictly increasing function for $\theta \geq 0$, we can see that the graphs will necessarily cross exactly once in each interval of the form

$$(n-1)\pi < \theta < \frac{(2n-1)}{2}\pi \quad (n = 1, 2, 3, \ldots)$$

or

$$\frac{(n-1)\pi}{2a} < \lambda_n < \frac{(2n-1)\pi}{2a} \quad (n = 1, 2, 3, \ldots).$$

□

Appendix 4

Theorem: *The function*

$$w[y,t] = \sum_{n=1}^{\infty} B_n \cos\lambda_n y e^{-\lambda_n^2 \frac{\kappa}{\rho c} t} \tag{1}$$

solves the heat equation

$$\frac{\partial^2 w}{\partial y} = \frac{\rho c}{\kappa}\frac{\partial w}{\partial y}; \quad 0 < y < a, \ t > 0. \tag{2}$$

Proof: Fix $t_1 > 0$. Then consider the series

$$\sum_{n=1}^{\infty} B_n \frac{d^k}{dy^k}\left[\cos\lambda_n y\right] e^{-\lambda_n^2 \frac{\kappa}{\rho c} t_1}, \quad k = 0, 1, 2. \tag{3}$$

We know that for all $n = 1, 2, 3, \ldots$

$$\frac{(n-1)\pi}{a} \le \lambda_n \le \frac{(2n-1)\pi}{2a}. \tag{4}$$

Therefore, for $k = 0, 1, 2$,

$$\lambda_n^k \le \left[\frac{(2n-1)\pi}{2a}\right]^k \text{ and } -\lambda_n^2 \le -\left[\frac{(n-1)\pi}{a}\right]^2. \tag{5}$$

We also know that for any *square integrable* function $g(y)$, B_n given by

$$B_n = \frac{\int_0^a g(y)\cos\lambda_n y \, dy}{\int_0^a \left(\cos\lambda_n y\right)^2 \, dy}$$

satisfies $B_n \to 0$ as $n \to \infty$. (See Tolstov, pp. 41–54, [8].) Thus, (4) and (5) imply for $k = 0, 1, 2$

$$\left|B_n \frac{d^k}{dy^k}\left[\cos\lambda_n y\right] e^{-\lambda_n^2 \frac{\kappa}{\rho c} t_1}\right| \le \left[\frac{(2n-1)\pi}{2a}\right]^k |B_n| e^{-\lambda_n^2 \frac{\kappa}{\rho c} t_1} =: M_{n,k}.$$

Now,

$$\frac{M_{n,k}}{1/n^2} = \left[\frac{(2n-1)\pi}{2a}\right]^k n^2 |B_n| e^{-\lambda_n^2 \frac{\kappa}{\rho c} t_1} = \frac{(2n-1)^k n^2 \pi^k}{2a e^{\lambda_n^2 \frac{\kappa}{\rho c} t_1}},$$

so applying *l'Hôpital's Rule* $k+2$ times we find

$$\lim_{n\to\infty} \frac{M_{n,k}}{1/n^2} = \lim_{n\to\infty} \frac{|B_n| 2^k (k+2)!}{(\text{some polynomial in } n) e^{\lambda_n^2 \frac{\kappa}{\rho c} t_1}} = 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p–series, $\sum_{n=1}^{\infty} M_{n,k}$ converges for $k = 0, 1, 2$. It follows from the *Weierstrass M–Test* [7] that the series in (3) converge uniformly for $k = 0, 1, 2$ on $0 < y < a$. Since $t = t_1$ was arbitrary, the series in (3) converge uniformly for all $0 < y < a$, for all $t > 0$. Therefore, Theorem 7.17 in Rudin [7] implies for all $0 < y < a$, $t > 0$,

$$\frac{\partial w}{\partial y}[y, t] = \sum_{n=1}^{\infty} -\lambda_n B_n \sin \lambda_n y e^{-\lambda_n^2 \frac{\kappa}{\rho c} t},$$

and

$$\frac{\partial^2 w}{\partial y}[y, t] = \sum_{n=1}^{\infty} -\lambda_n^2 B_n \cos \lambda_n y e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}.$$

Now fix $y_1 \in (0, a)$ and consider the series

$$\sum_{n=1}^{\infty} -\lambda_n^2 \frac{\kappa}{\rho c} B_n \cos \lambda_n y_1 e^{-\lambda_n^2 \frac{\kappa}{\rho c} t} \tag{6}$$

for $0 < \alpha \le t \le \beta$. It follows from (4) that for $0 < \alpha \le t \le \beta$,

$$\left[\frac{(n-1)\pi}{a}\right] \frac{\kappa}{\rho c} \alpha \le \lambda_n^2 \frac{\kappa}{\rho c} t \le \left[\frac{(2n-1)\pi}{2a}\right] \frac{\kappa}{\rho c} \beta. \tag{7}$$

Therefore (4) and (7) imply for $k = 0, 1$,

$$\left| \left[-\lambda_n^2 \frac{\kappa}{\rho c}\right]^k B_n \cos \lambda_n y_1 e^{-\lambda_n^2 \frac{\kappa}{\rho c} t} \right| \le \left[\frac{(2n-1)\pi}{2a} \frac{\kappa}{\rho c}\right]^k |B_n| e^{-\left(\frac{(n-1)\pi}{a}\right)^2 \lambda_n^2 \frac{\kappa}{\rho c} \alpha} = \tilde{M}_{n,k}.$$

Again we compare the series $\sum_{n=1}^{\infty} \tilde{M}_{n,k}$ ($k = 0, 1$) to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since l'Hôpital's Rule implies

$$\lim_{n\to\infty} \frac{\tilde{M}_{n,k}}{1/n^2} = \lim_{n\to\infty} \frac{|B_n| 2^k (k+2)! \left(\frac{\kappa}{\rho c}\right)^k}{(\text{some polynomial in } n) e^{-\left(\frac{(n-1)\pi}{a}\right)^2 \frac{\kappa}{\rho c} \alpha}} = 0,$$

the Weierstrass M–Test implies for $k = 0, 1$

$$\sum_{n=1}^{\infty} \left[-\lambda_n^2 \frac{\kappa}{\rho c}\right]^k B_n \cos \lambda_n y_1 e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}$$

converges uniformly on $0 < \alpha \le t \le \beta$ for any α, β fixed, for any $y_1 \in (0, a)$ fixed. Since y_1 is arbitrary, Theorem 7.17, [7], implies

$$\frac{\partial w}{\partial t}[y, t] = \sum_{n=1}^{\infty} -B_N \lambda_n^2 \frac{\kappa}{\rho c} \sin \lambda_n y e^{-\lambda_n^2 \frac{\kappa}{\rho c} t}$$

converge uniformly on any interval of the form $0 < \alpha \le t \le \beta$.

It follows that for $0 < y < a$, $0 < \alpha \le t \le \beta$, $w[y, t]$ solves the heat equation (2).

□

Appendix 5 - Demonstration *Mathematica* Code

■ Definition of Experimental Data Set

```
Tλ = { {2, 6.24}
, {4, 6.15}
, {6, 6.24}
, {8, 6.62}
, {10, 6.86}
, {12, 7.26}
, {14, 7.51}
, {16, 7.75}
, {18, 8.01}
, {20, 8.26}
, {22, 8.37}
, {24, 8.76}
, {26, 8.87}
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, {32, 9.37}
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, {38, 9.73}
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, {62, 11.07}
, {64, 11.19}
, {66, 11.32}
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, {78, 12.43}
, {80, 12.56}
, {82, 12.55}
, {84, 12.55}
, {86, 12.67}
, {88, 12.8}
, {90, 12.79}
, {92, 12.79}
, {94, 12.91}
```

```
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, {110, 13.14}
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, {116, 13.26}
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, {188, 13.59}
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, {192, 13.59}
, {194, 13.58}
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, {234, 13.56}
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, {238, 13.68}
, {240, 13.68}
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, {32, 3.04}
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, {50, 3.21}
```

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, {146, 4.3}
, {148, 4.3}
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```

```
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, {188, 4.8}
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, {192, 4.77}
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, {204, 4.9}
, {206, 5.04}
, {208, 5.04}
, {210, 5.04}
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, {214, 5.19}
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, {220, 5.16}
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, {226, 5.31}
, {228, 5.45}
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, {238, 5.43}
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};

Tc = { {2, 3.19}
, {4, 2.65}
, {6, 2.76}
```

```
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, {12, 2.67}
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, {18, 2.76}
, {20, 2.72}
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, {28, 2.95}
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, {52, 3.24}
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, {64, 3.33}
, {66, 3.3}
, {68, 3.47}
, {70, 3.44}
, {72, 2.82}
, {74, 3.41}
, {76, 3.41}
, {78, 3.38}
, {80, 3.38}
, {82, 3.53}
, {84, 3.36}
, {86, 3.5}
, {88, 3.5}
, {90, 3.65}
, {92, 3.47}
, {94, 3.47}
, {96, 3.62}
, {98, 3.62}
, {100, 3.77}
, {102, 3.23}
, {104, 3.74}
, {106, 3.74}
```

```
, {108, 3.89}
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, {112, 3.86}
, {114, 3.86}
, {116, 4.03}
, {118, 3.83}
, {120, 3.83}
, {122, 3.83}
, {124, 3.97}
, {126, 3.97}
, {128, 4.12}
, {130, 3.94}
, {132, 3.94}
, {134, 4.09}
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, {154, 4.18}
, {156, 4.33}
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, {160, 4.33}
, {162, 4.47}
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, {166, 4.3}
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, {172, 4.45}
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, {176, 4.42}
, {178, 4.56}
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, {184, 4.56}
, {186, 4.54}
, {188, 4.54}
, {190, 4.68}
, {192, 4.68}
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, {198, 4.66}
, {200, 4.66}
, {202, 4.8}
, {204, 4.8}
, {206, 4.8}
```

```
, {208, 4.8}
, {210, 4.95}
, {212, 4.95}
, {214, 4.77}
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, {218, 4.92}
, {220, 4.92}
, {222, 5.07}
, {224, 5.41}
, {226, 4.9}
, {228, 4.9}
, {230, 5.04}
, {232, 5.04}
, {234, 5.04}
, {236, 5.19}
, {238, 5.19}
, {240, 5.01}
};

Tmax = Ceiling[
   Max[Table[{TA[[i]][[2]], TB[[i]][[2]], TC[[i]][[2]]}, {i, 1, Length[TA]}]]];

PA = ListPlot[TA, PlotRange -> {0, Tmax}, PlotStyle -> Hue[0],
   AxesLabel -> {"t", "U"}, PlotLabel → "Initial Experimental U[a,t]"];
PB = ListPlot[TB, PlotRange -> {0, Tmax}, PlotStyle -> Hue[0.4],
   AxesLabel -> {"t", "U"}, PlotLabel → "Initial Experimental U[a/2,t]"];
PC = ListPlot[TC, PlotRange -> {0, Tmax}, PlotStyle -> Hue[0.6],
   AxesLabel -> {"t", "U"}, PlotLabel → "Initial Experimental U[0,t]"];
Show[PA, PB, PC, PlotLabel → "Initial Experimental U[y,t]"];
```

■ Definition of Constants

```
T0 = 21;
a = 28;
ρ = 1;
c = 1;
maxterms = 30;
κ = 0.00144 * 60;
h = 0.002 * 11;
```

■ Getting initial temperature data

```
TminA = TA[[1]][[2]];
TminB = TB[[1]][[2]];
TminC = TC[[1]][[2]];
```

■ Definition of f[y] - Piece-wise Linear

```
Clear[f];
f[y_] := ((TminB - TminC)/(a/2)) y + TminC /; 0 ≤ y ≤ a/2;
f[y_] := ((TminA - TminB)/(a/2)) (y - a/2) + TminB /; a/2 < y ≤ a;
Show[Plot[f[y], {y, 0, a}, PlotRange -> {0, Tmax}],
  ListPlot[{{a, TA[[1]][[2]]}, {a/2, TB[[1]][[2]]}, {0, TC[[1]][[2]]}},
   PlotRange -> {0, Tmax}, PlotStyle → PointSize[0.02]],
  PlotLabel → "Initial Temperature Distribution", AxesLabel → {"y", "U"}];
```

■ Obtaining Values for λ_n

(These values (within working precision) could also be obtained using *Mathematica*'s FindRoot but FindRoot doesn't always converge. To avoid this problem, we used a for-loop to obtain an approximation using the Bisection Method to find the λ_n values.)

```
Clear[λ]
λ[n_] := For[
    x1 = (n - 1) π / a; v1 = κ x1 Sin[a x1] - h Cos[a x1];
    x2 = (2 n - 1) π / (2 a); v2 = κ x2 Sin[a x2] - h Cos[a x2];
    xt = (x1 + x2)/2; vt = κ xt Sin[a xt] - h Cos[a xt],
Abs[x2 - x1] > 10^-$MachinePrecision,
    xt = (x1 + x2)/2; vt = κ xt Sin[a xt] - h Cos[a xt],
    If[vt > 0,
        If[v1 > 0, x1 = xt, x2 = xt],
     If[v1 < 0, x1 = xt, x2 = xt]];
    If[Abs[x2 - x1] ≤ 10^-$MachinePrecision, λ[n] = N[xt]]
    ];
Table[λ[n], {n, 1, maxterms}];
```

■ Obtaining Values for B_n

```
Table[B[n] = Re[∫_0^a (f[y] - T0) Cos[λ[n] y] dy / ∫_0^a Cos[λ[n] y]^2 dy], {n, 1, maxterms}];
```

- **When is enough enough?**

```
For[maxn = 5, maxn ≤ maxterms, maxn = maxn + 5,
 u[y_, t_] = T0 + Sum[B[n] Cos[λ[n] y] E^(-λ[n]^2 κ/(ρ σ) t), {n, 1, maxn}]; U[y_, t_] = N[u[y, t]];
 Plot[{f[y], U[y, 0]}, {y, 0, a}, PlotRange → {0, Tmax},
  PlotStyle → {Hue[0], Hue[0, 0, 0]}, PlotLabel → {maxn "=n"}, AxesLabel → {"t", "U"}]]
```

- **Behavior of Model**

```
maxn = 30;
u[y_, t_] = T0 + Sum[B[n] Cos[λ[n] y] E^(-λ[n]^2 κ/(ρ σ) t), {n, 1, maxn}];
U[y_, t_] = N[u[y, t]];
Show[
  Plot[{U[0, t], U[a/2, t], U[a, t]}, {t, 0, 4 * 60}, PlotRange -> {0, Max[U[a, 4 * 60], Tmax]}
   PlotStyle → {Hue[0.6], Hue[.4], Hue[0]}, AxesLabel → {"t", "U"},
   PlotLabel → "Model vs Initial \n Experimental U[y,t]"], PA, PB, PC];
```

■ Verifying Our Model with New Data

■ Definition of New Experimental Data Set

```
T₁ =
{ {2, 3.18}
, {4, 3.09}
, {6, 3.75}
, {8, 4.11}
, {10, 4.47}
, {12, 4.68}
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, {16, 5.4}
, {18, 5.63}
, {20, 5.98}
, {22, 6.07}
, {24, 6.3}
, {26, 6.53}
, {28, 6.76}
, {30, 6.99}
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, {38, 7.53}
, {40, 7.76}
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, {44, 7.98}
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, {50, 8.43}
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```

```
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, {134, 10.79}
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```
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};

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, {104, 1.34}
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, {108, 1.34}
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, {136, 1.71}
, {138, 1.71}
, {140, 1.88}
```

```
, {142, 1.88}
, {144, 1.88}
, {146, 1.68}
, {148, 1.84}
, {150, 1.84}
, {152, 1.84}
, {154, 1.64}
, {156, 2}
, {158, 1.81}
, {160, 1.81}
, {162, 1.81}
, {164, 1.81}
, {166, 1.97}
, {168, 1.97}
, {170, 1.97}
, {172, 1.97}
, {174, 2.13}
, {176, 1.93}
, {178, 1.93}
, {180, 1.93}
, {182, 1.93}
, {184, 1.9}
, {186, 2.09}
, {188, 2.09}
, {190, 2.26}
, {192, 2.06}
, {194, 2.06}
, {196, 2.06}
, {198, 2.22}
, {200, 2.22}
, {202, 2.22}
, {204, 2.38}
, {206, 2.18}
, {208, 2.18}
, {210, 2.35}
, {212, 2.35}
, {214, 2.51}
, {216, 2.31}
, {218, 2.31}
, {220, 2.47}
, {222, 2.47}
, {224, 2.47}
, {226, 2.47}
, {228, 2.64}
, {230, 2.64}
, {232, 2.64}
, {234, 2.6}
, {236, 2.44}
, {238, 2.6}
, {240, 2.6}
```

```
};

Tc = { {2, 1.06}
, {4, 0.87}
, {6, 0.68}
, {8, 0.84}
, {10, 0.8}
, {12, 0.61}
, {14, 0.57}
, {16, 0.73}
, {18, 0.69}
, {20, 0.65}
, {22, 0.65}
, {24, 0.62}
, {26, 0.58}
, {28, 0.58}
, {30, 0.54}
, {32, 0.5}
, {34, 0.5}
, {36, 0.47}
, {38, 0.62}
, {40, 0.43}
, {42, 0.59}
, {44, 0.39}
, {46, 0.55}
, {48, 0.55}
, {50, 0.51}
, {52, 0.51}
, {54, 0.47}
, {56, 0.63}
, {58, 0.44}
, {60, 0.44}
, {62, 0.6}
, {64, 0.4}
, {66, 0.56}
, {68, 0.56}
, {70, 0.52}
, {72, 0.52}
, {74, 0.68}
, {76, 0.48}
, {78, 0.64}
, {80, 0.64}
, {82, 0.44}
, {84, 0.44}
, {86, 0.6}
, {88, 0.4}
, {90, 0.56}
, {92, 0.96}
, {94, 0.36}
, {96, 0.53}
```

```
, {98, 0.53}
, {100, 0.53}
, {102, 0.49}
, {104, 0.49}
, {106, 0.49}
, {108, 0.65}
, {110, 0.45}
, {112, 0.45}
, {114, 0.45}
, {116, 0.61}
, {118, 0.61}
, {120, 0.41}
, {122, 0.58}
, {124, 0.58}
, {126, 0.58}
, {128, 0.58}
, {130, 0.53}
, {132, 0.53}
, {134, 0.53}
, {136, 0.53}
, {138, 0.7}
, {140, 0.5}
, {142, 0.5}
, {144, 0.5}
, {146, 0.66}
, {148, 0.66}
, {150, 0.66}
, {152, 0.66}
, {154, 0.62}
, {156, 0.62}
, {158, 0.62}
, {160, 0.62}
, {162, 0.78}
, {164, 0.58}
, {166, 0.58}
, {168, 0.58}
, {170, 0.58}
, {172, 0.75}
, {174, 0.75}
, {176, 0.55}
, {178, 0.71}
, {180, 0.71}
, {182, 0.71}
, {184, 0.71}
, {186, 0.87}
, {188, 0.87}
, {190, 0.67}
, {192, 0.67}
, {194, 0.67}
, {196, 0.83}
```

```
, {198, 0.83}
, {200, 0.83}
, {202, 0.63}
, {204, 0.63}
, {206, 0.79}
, {208, 0.79}
, {210, 0.79}
, {212, 0.79}
, {214, 0.96}
, {216, 0.96}
, {218, 0.76}
, {220, 0.76}
, {222, 0.92}
, {224, 0.92}
, {226, 0.92}
, {228, 0.92}
, {230, 1.08}
, {232, 0.88}
, {234, 0.88}
, {236, 1.05}
, {238, 1.05}
, {240, 1.05}
};

Tmax = Ceiling[
   Max[Table[{TA[[i]][[2]], TB[[i]][[2]], TC[[i]][[2]]}, {i, 1, Length[TA]}]]];

PA = ListPlot[TA, PlotRange -> {0, Tmax}, PlotStyle -> Hue[0],
   AxesLabel -> {"t", "U"}, PlotLabel → "New Experimental U[a,t]"];
PB = ListPlot[TB, PlotRange -> {0, Tmax}, PlotStyle -> Hue[0.4],
   AxesLabel -> {"t", "U"}, PlotLabel → "New Experimental U[a/2,t]"];
PC = ListPlot[TC, PlotRange -> {0, Tmax}, PlotStyle -> Hue[0.6],
   AxesLabel -> {"t", "U"}, PlotLabel → "New Experimental U[0,t]"];
Show[PA, PB, PC, PlotLabel → "New Experimental U[y,t]"];
```

▪ Definition of New Constants

```
T0 = 24;
a = 28;
ρ = 1;
c = 1;
maxterms = 30;
κ = 0.00144 * 60 * 3;
h = 0.00103 * 23;
```

- **Getting temperature minima**

```
T_minA = Min[Table[T_A[[i]][[2]], {i, 1, Length[T_A]}]];
T_minB = Min[Table[T_B[[i]][[2]], {i, 1, Length[T_B]}]];
T_minC = Min[Table[T_C[[i]][[2]], {i, 1, Length[T_C]}]];
```

- **Defining f[y] of the form $A\,e^{B\,y} + C$**

```
Clear[f];
f[y_] = (T_minC + (T_minB^2 - T_minA T_minC) / (T_minA - 2 T_minB + T_minC)) * Exp[((2 Log[(T_minB - T_minA)/(T_minC - T_minB)])/a) * y] +
    ((T_minA T_minC - (T_minB)^2) / (T_minA - 2 T_minB + T_minC));
Show[Plot[f[y], {y, 0, a}, PlotRange -> {0, T_max}],
  ListPlot[{{a, T_A[[1]][[2]]}, {a/2, T_B[[1]][[2]]}, {0, T_C[[1]][[2]]}},
   PlotRange -> {0, T_max}, PlotStyle -> PointSize[0.02]],
  PlotLabel → "Alternate Temperature Distribution", AxesLabel → {"y", "U"}];
```

- **Obtaining Alternate Values for λ_n**

```
Clear[λ]
λ[n_] := For[
    x_1 = ((n - 1) π)/a; v_1 = κ x_1 Sin[a x_1] - h Cos[a x_1];
    x_2 = ((2 n - 1) π)/(2 a); v_2 = κ x_2 Sin[a x_2] - h Cos[a x_2];
    x_t = (x_1 + x_2)/2; v_t = κ x_t Sin[a x_t] - h Cos[a x_t],
Abs[x_2 - x_1] > 10^-$MachinePrecision,
    x_t = (x_1 + x_2)/2; v_t = κ x_t Sin[a x_t] - h Cos[a x_t],
    If[v_t > 0,
        If[v_1 > 0, x_1 = x_t, x_2 = x_t],
     If[v_1 < 0, x_1 = x_t, x_2 = x_t]];
    If[Abs[x_2 - x_1] ≤ 10^-$MachinePrecision, λ[n] = N[x_t]]
    ];
Table[λ[n], {n, 1, maxterms}];
```

- **Obtaining Alternate Values for B_n**

```
Table[B[n] = Re[(∫_0^a (f[y] - T_0) Cos[λ[n] y] dy)/(∫_0^a Cos[λ[n] y]^2 dy)], {n, 1, maxterms}];
```

■ Behavior of Model with Alternate f[y]

```
maxn = 30;
```

$$u[y\_,\ t\_] = T_0 + \sum_{n=1}^{maxn} B[n]\ Cos[\ \lambda[n]\ y]\ E^{-\lambda[n]^2 \frac{\kappa}{\rho c} t};$$

```
U[y_, t_] = N[u[y, t]];
```

$$Show[$$
$$\quad Plot\left[\left\{U[0,\ t],\ U\left[\frac{a}{2},\ t\right],\ U[a,\ t]\right\},\ \{t,\ 0,\ 4*60\},\ PlotStyle \to \{Hue[0.6],\ Hue[.4],\ Hue[0]$$
$$\quad AxesLabel \to \{"t",\ "U"\},\ PlotRange \to \{0,\ Max[U[a,\ 4*60],\ T_{max}]\},$$
$$\quad PlotLabel \to "Alternative\ Model\ vs\ New\ \backslash n\ Experimental\ U[y,t]"\right],\ P_A,\ P_B,\ P_C];$$

■ The Bottom Line for the Alternate f[y]

```
maxn = 10;
```

$$u[y\_,\ t\_] = T_0 + \sum_{n=1}^{maxn} B[n]\ Cos[\ \lambda[n]\ y]\ E^{-\lambda[n]^2 \frac{\kappa}{\rho c} t};$$

```
U[y_, t_] = N[u[y, t]];
```

$$AveU[t\_] = \frac{\int_0^a U[y,\ t]\ dy}{a};$$

```
Plot[{15, AveU[t]}, {t, 0 * 24 * 60, 5 * 24 * 60}, PlotStyle -> {Hue[0], RGBColor[0, 0, 0]},
  AxesLabel -> {"t", "Ave U"}, PlotLabel -> "Average Temperature and Cold"];
```

By inspection, we can pick a reasonable starting point and let *Mathematica*'s FindRoot give us the bottom line.

```
Re[(t /. FindRoot[AveU[t] - 15 == 0, {t, 1.5 * 24 * 60}]) / 60 / 24]
```